# Spatio-Temporal Pricing for Ridesharing Platforms*

Hongyao Ma † Fei Fang ‡ David C. Parkes §

**Abstract**

Ridesharing platforms match drivers and riders to trips, using dynamic prices to balance supply and demand. A challenge is to set prices that are appropriately smooth in space and time, so that drivers will choose to accept their dispatched trips, rather than drive to another area or wait for higher prices or a better trip. We work in a complete information, discrete time, multi-period, multi-location model, and introduce the *Spatio-Temporal Pricing (STP) mechanism*. The mechanism is incentive-aligned, in that it is a subgame-perfect equilibrium for drivers to accept their dispatches, and the mechanism is also welfare-optimal, envy-free, individually rational and budget balanced from any history onward. The proof of incentive alignment makes use of the $M^\natural$ concavity of min-cost flow objectives. We also give an impossibility result, that there can be no dominant-strategy mechanism with the same economic properties. An empirical analysis conducted in simulation suggests that the STP mechanism can achieve significantly higher social welfare than a myopic pricing mechanism.

## 1 Introduction

Ridesharing platforms such as Uber and Lyft are disrupting traditional forms of transit. These are two-sided platforms, with both riders and drivers in a customer relationship with the platform. When a rider opens the app and enters an origin and destination, these platforms quote a price for the trip and an estimated wait time until a driver will arrive. If a rider requests the ride, the platform offers the pickup opportunity to each of a sequence of nearby drivers until a driver accepts. At this point, if neither side cancels and the driver completes the pick-up then the trip begins. Once the trip is complete, payment is made from the rider to the driver through the platform (with some cut of the payment going to the platform).[1]

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†John A. Paulson School of Engineering and Applied Sciences, Harvard University, 33 Oxford Street, Maxwell Dworkin 242, Cambridge, MA 02138, USA. Email: hma@seas.harvard.edu.

‡School of Computer Science, Carnegie Mellon University, Hamerschlag Dr, 4126 Wean Hall, Pittsburgh, PA 15213, USA. Email: feifang@cmu.edu.

§John A. Paulson School of Engineering and Applied Sciences, Harvard University, 33 Oxford Street, Maxwell Dworkin 229, Cambridge, MA 02138, USA. Email: parkes@eecs.harvard.edu.

[1] The actual practice is somewhat more complicated, in that platforms may operate multiple products, for example, high-end cars, sports utility vehicles, trips shared by multiple riders, etc. Moreover, drivers within even a single class are differentiated (e.g., cleanliness of car, skill of driving), as are riders (e.g., politeness, loud vs. quiet, safety of neighborhood). We ignore these effects in our model, and assume that, conditioned on the same trip, all drivers are equivalent from the perspective of riders and all riders equivalent from the perspective of drivers.

Several ridesharing platforms emphasize the importance of providing reliable transportation. For example, Uber's mission is "to connect riders to reliable transportation, everywhere for everyone."[2] This mission is also stated as "to make transportation as ubiquitous and reliable as running water."[3] Lyft's mission is "to provide the best, most reliable service possible by making sure drivers are on the road when and where you need them most."[4] Whereas taxi systems have reliable pricing but unreliable service [31], these platforms make use of dynamic pricing to achieve reliable services. They also emphasize the flexibility for drivers to drive on their own schedule. Uber advertises itself as "work that put you first— drive when you want, earn what you need,"[5] and Lyft promises drivers "To drive or not to drive? It's really up to you."[6]

Despite their success, there remain a number of problems with the design of the rules that govern these ridesharing platforms, leading in turn to various kinds of market failure. A particular concern, is that trips may be mis-priced relative to each other— in comparison to always accepting the platform's dispatch, drivers may earn a higher income by strategically canceling particular trips [18, 13]. Imposing cancellation penalties on drivers incentivizes them to go offline and choose not to participate in the platform from certain locations or times, and as a result introduces further challenges on the correct estimation of driver supply. In this way, poorly designed pricing and dispatching systems undercut the ability of these platforms to fulfill their mission of reliable transport while providing drivers with the flexible spirit of the gig economy.[7]

One kind of mis-pricing is spatial. Consider, for example, that if the price is substantially higher for trips that start in location $A$ than an adjacent location $B$, drivers in location $B$ that are close to the boundary will decline trips. This spatial mis-pricing leads to drivers "chasing the surge"— turning off a ridesharing app while relocating to another location where prices are higher.[8] This results in a loss in reliability and rider welfare, with even high willingness-to-pay riders in location $B$ unable to access reliable transportation.

Problems with spatial pricing also arise because prices do not correctly factor market conditions at the destination of a trip. It has been standard practice to use origin-based, dynamic "surge pricing", with unit prices that depend on market conditions at the origin but not on market conditions at the destination. Suppose that a driver in location $A$ could be dispatched to a trip to a quiet suburb $B$ or a trip to the downtown area $C$, both trips taking the same time and covering the same distance. With origin-based pricing, the continuation payoff for the drive in $B$ is expected to be smaller than that in $C$, as higher demand in $C$ would lead to a lower wait time and higher prices. But under origin-based pricing, the payments for these trips would be the same, incentivizing drivers to decline trips to the suburbs, resulting in a market failure with even high willingness-to-pay $(A, B)$ riders unable to access reliable transportation.[9]

A second kind of mis-pricing is temporal. Consider a setting where a sports event will end soon, and drivers can anticipate that prices will increase in order to balance supply and demand. In this

[2] `https://www.uber.com/legal/community-guidelines/us-en/`, visited September 1, 2017.

[3] `http://time.com/time-person-of-the-year-2015-runner-up-travis-kalanick/`, visited September 1, 2017.

[4] `https://help.lyft.com/hc/en-us/articles/115012926227`, visited September 1, 2017.

[5] `https://www.uber.com/drive/`, visited December 12, 2017.

[6] `https://blog.lyft.com/posts/reasons-to-drive-with-lyft`, visited December 12, 2017.

[7] There are also other incentive problems, including inconsistencies across classes of service, competition among platforms, and drivers' off-platform incentives. In the interest if simplicity, we only model a single class of service, ignore cross-platform competition, and do not model location- or time-dependent opportunity cost.

[8] `http://maximumridesharingprofits.com/advice-new-uber-drivers-dont-chase-surge/`, visited Jan 8, 2018.

[9] This partially explains the phenomenon of drivers calling riders to ask for the destination of a trip, as well as why platforms hide the destination from a driver before a trip begins. `https://maximumridesharingprofits.com/cant-uber-drivers-see-passengers-destination-accepting-trip/`, visited November 18, 2017.

case, drivers will decline trips in anticipation of the price surge. This results in a market failure, with even high willingness-to-pay riders unable to get picked up before the end of the event.

We can conceptualize these kinds of mis-pricing as a failure of prices to be appropriately "smooth" in space and time— if prices for trips are higher in one location then they should be appropriately higher in adjacent locations; if destinations differ in continuation payoffs then trip prices to these destinations need to reflect this; and if demand would soon increase in a location then the current prices should already be appropriately higher. A good working definition of smoothness is that drivers who retain the flexibility to choose when to work will always choose to accept any trip to which they are dispatched (recall that driver, too, are in a customer relationship with the platform). In this way, smooth prices are responsive to a central challenge of market design for ridesharing platforms, which is to optimally orchestrate trips without the power to tell drivers what to do.[10]

Another concern relates to the fairness and equity of the platform in regard to drivers. This has arisen for example in the context of the gender gap in hourly earnings [18], and weekly bonuses that may be based on the number of trips completed by a driver.[11] Recognizing these kinds of fairness concerns, an additional desirable property of ridesharing markets is that a pair of drivers in the same location at the same time do not envy each others' future income. This removes the arbitrariness of dispatch decisions, where drivers are not dependent on lucky dispatches to gain competitive earnings relative to their peers. More broadly, we can expect that providing equity in long-run earnings can promote ongoing participation by drivers on a platform.

### 1.1 Our results

We propose a new mechanism for dispatching and pricing in the context of a ridesharing platform, addressing the problem of incentive alignment for drivers even in the absence of time-extended contracts (which we view as incompatible with the spirit of the gig economy). In particular, we propose the *Spatio-Temporal Pricing (STP) ridesharing mechanism*, under which accepting the mechanism's dispatches at all times forms a subgame-perfect equilibrium (SPE) among the drivers. From any history onward, the STP mechanism is individually rational, budget balanced, welfare optimal, and also envy-free, meaning that any pair of drivers in the same location at the same time has the same continuation payoff.[12] We also give an impossibility result, that there can be no dominant-strategy mechanism with the same economic properties, and show via simulation that the STP mechanism achieves significantly higher social welfare than a myopic pricing mechanism.

We work in a complete information, discrete time, multi-period, multi-location model, and allow asymmetric, time-varying trip times, non-stationarity in demand, and riders with different values for completing a trip. At the beginning of each time period, based on the history, current positioning of drivers, and current and future demand, the STP mechanism dispatches a trip to each available driver (including the possibility of relocating a driver), and determines a payment to be made if the driver follows the dispatch. Each driver can decide whether to follow the suggested action, or to decline and stay or relocate to any location (without getting paid). After observing the

[10] There is an echo here to *market-oriented programming* [34] and *agoric systems* [16]. There, markets with virtual prices were suggested as a means for achieving optimization in decentralized systems. But whereas this earlier work adopted market-based methods for their ability to optimize, and prices were virtual, there is an additional need in the present setting to align incentives.

[11] `https://uberpeople.net/threads/ubers-newest-way-to-screw-us-over-incentive-bonuses.146416/`, visited January 9, 2018.

[12] The STP mechanism balances budget. Alternatively, we many think about the ridesharing platform taking a fixed percentage of the payments. This does not affect the results presented in this paper.

driver actions in a period, the mechanism collects payments from the riders and makes payments to the drivers.

We allow for heterogeneity in rider values and trip details, as well as in the location and time at which different drivers enter the platform. The main assumptions that we make are (i) complete information about supply and demand over a planning horizon, (ii) impatient riders that need to be picked-up at a particular time and location (and without preferences over drivers), and (iii) drivers who are willing to take trips until the end of the planning horizon, and with no intrinsic preference for locations or passengers, and no heterogeneity in costs. We model a rider's value as her willingness-to-pay, over-and-above a base payment that is charged for trips, and set by the platform so that completed trips in aggregate cover driver costs. In this way, we do not explicitly model driver costs, and the prices determined by the STP mechanism can be considered to correspond to the "surge price," that is the payment that is collected over-and-above the base payment for a trip.

We prove the existence of anonymous, origin-destination, competitive equilibrium (CE) prices, allowing the unit price of a trip can depend on market conditions at both the origin and destination.[13] The STP mechanism users *driver-pessimal CE prices*, recomputing a driver-pessimal CE plan in the event of driver deviations from the current plan. The mechanism induces an extensive-form game among the drivers, where the total payoff to each driver is determined by the mechanism's dispatch and payment rules. Somewhat surprisingly, the use of driver-pessimal CE prices (vs., for example, driver-optimal CE prices as in Vickrey-Clarke-Groves mechanisms) is an essential part of achieving our results. The proof of incentive alignment makes use of the $M^\natural$ concavity of min-cost flow objectives [25]. The same connection to min-cost flow leads to an efficient algorithm to compute an optimal dispatch plan and prices, and to operationalize the STP mechanism.

The rest of this paper is organized as follows. After a brief discussion on related work, we introduce the model in Section 2, and illustrate through an example that a myopic pricing mechanism, which naively clears the market for each location without considering future demand and supply, fails to be welfare-optimal or incentive aligned. In Section 3, we formulate the optimal planning problem and establish integrality properties of a linear-programming relaxation (Lemma 2). We show that a plan with anonymous trip prices is welfare-optimal if and only if it forms a competitive equilibrium (CE) (Lemma 3), and that optimal CE plans exist and are efficient to compute. We also prove that drivers' total payments among all CE plans form a lattice (Lemma 4). A class of static CE mechanisms are discussed, that incentivize drivers to always follow the mechanism's dispatches (Theorem 1) but fail to be welfare-optimal or envy-free after driver deviations.

We prove our main result in Section 4, that the STP mechanism is subgame-perfect incentive compatible, and is also individually rational, budget balanced, envy-free, and welfare optimal from any history onward (Theorem 2). We also provide an impossibility result, that no dominant-strategy mechanism has the same economic properties (Theorem 3), followed by discussions on the effect of relaxing the model assumptions. An empirical analysis conducted through simulation (Section 5) suggests that the STP mechanism can achieve significantly higher social welfare than a myopic pricing mechanism, and highlights the failure of incentive alignment due to non-smooth prices in

[13]The idea that the unit price for a trip might depend on the destination is already familiar in taxi systems, with trips to the suburbs billed with a surcharge to reflect that drivers may need to return to the city without a fare. Origin-destination pricing also seems practical. Indeed, ridesharing platforms are moving in this direction, and this is facilitated by the movement to quoting a total payment for a trip rather than an origin-based surge multiplier. `https://newsroom.uber.com/upfront-fares-no-math-and-no-surprises/`, visited September 1, 2017; `https://blog.lyft.com/posts/now-live-see-your-ride-total`, visited September 1, 2017; `https://www.bloomberg.com/news/articles/2017-05-19/uber-s-future-may-rely-on-predicting-how-much-you-re-willing-to-pay`, visited September 1, 2017.

myopic mechanisms. We consider three stylized scenarios: the end of a sports event, the morning rush hour, and trips to and from the airport with imbalanced flows. In all three scenarios, the STP mechanism achieves substantially higher social welfare as well as time-efficiency for drivers. We conclude in Section 6. Additional examples, omitted proofs, simulation results and relations to the literature are provided in the Appendix.

### 1.2 Related Work

To the best of our knowledge, this current paper is unique in that it considers both multiple locations and multiple time periods, along with rider demand, rider willingness-to-pay, and driver supply that can vary across both space and time. This leads to the focus of the present paper on the design of a ridesharing mechanism with prices that are smooth in both time and space.

Earlier, Banerjee et al. [5] adopt a queuing-theoretic approach in analyzing the effect of dynamic pricing on the revenue and throughput of ridesharing platforms, assuming a single location and stationary system state. In this context, the optimal dynamic pricing strategy, where prices can depend on supply and demand conditions, does not achieve better performance than the optimal static pricing strategy when the platform correctly estimates supply and demand. However, dynamic pricing is more robust to fluctuations and to mis-estimation of system parameters. By analizing a two-location, stationary state queueing model, Afèche et al. [1] study the impact of platform control on platform revenue and driver income.

By analyzing the equilibrium outcome under a continuum model (supply and demand), and with stationary demand and unlimited driver supply (at fixed opportunity costs), Bimpikis et al. [9] in independent, contemporaneous work study the steady-state and show that a ridesharing platform's profit is maximized when the demand pattern across different locations is balanced. In simulation they show that, in comparison to setting a single price, pricing trips differently depending on trip origins improves the platform's profit. The same simulation also shows that there is not a substantial, additional gain from using origin-destination based pricing in their model. Our model is quite distinct, in that it is not a continuum model, does not have unlimited driver supply, and is not stationary. These authors explain that the solution to optimal origin-destination prices in their model only has dimensionality linear in the number of locations, but do not offer a complete theory to explain their simulation results. Banerjee et al. [4], also independent and contemporaneous, model a shared vehicle system as a continuous-time Markov chain, and establish approximation guarantees for a static, state-independent pricing policy (i.e. fixed prices that do not depend on the spatial distribution of cars), w.r.t. the optimal, state-dependent policy.

Castillo et al. [11] study the impact of myopically dispatching the closest drivers to rider requests. In particular, they discuss a "wild goose chase" phenomena. When demand much exceeds supply, myopic dispatching means that drivers spend too much time driving to pick up riders instead of carrying riders on trips, which leads to decreased social welfare and revenue to the platform. The theoretical model assumes a stationary state, where driver supply is driven by hourly earnings, rider demand depends on trip prices and wait times, and the wait time increases as the number of idle cars decreases. They do not have an explicit model of location. Empirical evidence is provided by analyzing Uber data. The wild goose chase is an effect of the ridesharing platform always dispatching a driver as soon as any rider requests a ride. The authors establish the importance of dynamic pricing when using this kind of myopic dispatching scheme, in keeping enough cars idle to avoid the inefficiency of long pick-ups, and show this is superior to other possible solutions, for example limiting the pick-up radius.

There are various empirical studies of the Uber platform as a two-sided marketplace [20, 21, 17], analyzing Uber's driver partners, the labor market equilibrium and consumer surplus. By analyzing the hourly earnings of drivers on the Uber platform, Chen et al. [14] show that drivers' reservation wages vary significantly over time, and that the real-time flexibility of being able to choose when to work increases both driver surplus and the supply of drivers; Cook et al. [18] show that driving speed, preferred time and location to drive, driver experience and their ability to strategically canceling rides together contribute to a 7% gender gap in hourly earnings. In regard to dynamic pricing, Chen and Sheldon [15] show, by analyzing the trips provided by a subset of driver partners in several US cities from 2014-2015, that surge pricing increases the supply of drivers on the Uber platform at times when the surge pricing is high. A case study [19] into an outage of Uber's surge pricing during the 2014-2015 New Year's Eve in New York City found a large increase in riders' waiting time after requesting a ride, and a large decrease in the percentage of requests completed.

The dynamic variations on the VCG mechanism [3, 6, 12] truthfully implement efficient decision policies, where agents receive private information over time. These mechanisms are not suitable for our problem, however, because some drivers may be paid negative payments for certain periods of time. The payment to an agent in a single period in the dynamic VCG mechanism is equal to the flow marginal externality imposed on the other agents by its presence in the current period [12]. The problem in our setting is that the existence of a driver for only one period may exert negative externality on the rest of the economy by inducing suboptimal positioning of the rest of the drivers in the subsequent time periods. See Appendix D.1 for examples and discussions.

The literature on *trading networks* studies economic models where agents in a network can trade via bilateral contracts [22, 23, 29]. Efficient, competitive equilibrium outcomes exist when agents' valuation functions satisfy the "full substitution" property, and the utilities of agents on either end of an acyclic network form lattices. Under proper assumptions, the optimal dispatching problem in ridesharing can be reduced to a trading network problem, where drivers and riders trade the right to use a car for the rest of the planning horizon (see Appendix D.2). However, the underlying network in the ridesharing problem evolves as time progresses, and to our knowledge, the dynamics and incentives in trading networks with time-varying networks are not studied in the literature.

Principal-agent problems have been studied extensively in contract theory [10, 32], where problems with information asymmetry before the time of contracting are referred to as *adverse selection*, and problems where asymmetric information or hidden action arise after the time of contracting are referred to as *moral hazard*. Contracts specify how agents are going to be rewarded or penalized based on observed performance measures. In the setting where contracts cannot be perfectly enforced, *relational incentive contracts* [24] have also been studied, which are self-enforcing by threatening to terminate an agent following poor performance. In our model, there are neither hidden actions nor asymmetric information. Instead, the challenge we address is one of incentive alignment in the absence of contracts. We insist on retaining flexibility for drivers, in the spirit of the customer relationship with platforms and the Gig economy, with drivers free to decide on their own actions without incurring penalties or termination threats.

## 2 Preliminaries

Let $T$ be the length of the planning horizon, starting at time $t = 0$ and ending at time $t = T$. We adopt a discrete time model, and refer to each time point $t$ as "time $t$", and call the duration between time $t$ and time $t + 1$ a *time period* or a *unit of time*. Trips start and end at time points. We may think about each time period as $\sim 5$ minutes, and with $T = 6$ the planning horizon would be half an hour. Denote $[T] = \{0, 1, \ldots, T\}$ and $[T - 1] = \{0, 1, \ldots, T - 1\}$.

Let $\mathcal{L} = \{A, B, \ldots, \}$ be a set of $|\mathcal{L}|$ discrete locations, and we adopt $a$ and $b$ to denote generic locations. For all $a, b \in \mathcal{L}$ and $t \in [T]$, the triple $(a, b, t)$ denotes a *trip* with origin $a$, destination $b$, starting at time $t$. Each trip can represent (i) taking a rider from $a$ to $b$ at time $t$, (ii) relocating without a rider from $a$ to $b$ at time $t$, and (iii) staying in the same location for one period of time (in which case $a = b$). Let $\delta : \mathcal{L} \times \mathcal{L} \to \mathbb{N}$ denote the time to travel between locations, so that trip $(a, b, t)$ ends at $t + \delta(a, b)$.[14] We allow $\delta(a, b) \neq \delta(b, a)$ for locations $a \neq b$, modeling asymmetric traffic flows. We assume $\delta(a, b) \geq 1$ and $\delta(a, a) = 1$ for all $a, b \in \mathcal{L}$, and that the triangle inequality holds.[15] Let $\mathcal{T} = \{ (a, b, t) \mid a \in \mathcal{L},\ b \in \mathcal{L},\ t \in \{0, 1, \ldots, T - \delta(a, b)\}\}$ denote the set of all feasible trips within the planning horizon.

Let $\mathcal{D}$ denote the set of drivers, with $m \triangleq |\mathcal{D}|$. Each driver $i \in \mathcal{D}$ is characterized by *type* $\theta_i = (\underline{\tau}_i, \bar{\tau}_i, \ell_i)$— driver $i$ enters the platform at time $\underline{\tau}_i$ and location $\ell_i$, and plans to exit the platform at time $\bar{\tau}_i$ (with $\underline{\tau}_i < \bar{\tau}_i$). Here we make the assumption (S1) that *driver types are known to the mechanism and that all drivers stay until at least the end of planning horizon, and do not have an intrinsic preference over location, including where they finish their last trip in the planning horizon.* Each driver seeks to maximize the total payment received over the planning horizon.

Denote $\mathcal{R}$ as the set of riders, each intending to take a single trip during the planning horizon. The *type* of rider $j \in \mathcal{R}$ is $(o_j, d_j, \tau_j, v_j)$, where $o_j$ and $d_j$ are the trip origin and destination, $\tau_j$ the requested start time, and $v_j \geq 0$ the *value* for the trip. We assume (S2) *that riders are impatient, only value trips starting at $\tau_j$, are not willing to relocate or walk from a drop-off point to their actual, intended destination.* The value $v_j$ models the willingness-to-pay of the rider, over-and-above a base payment for a trip. This base payment is an amount that is collected by the platform, so that in aggregate drivers' costs are covered. Accordingly, the prices we derive in this paper correspond to surge prices, over-and-above these base payments.[16] Rider utility is quasi-linear, with utility $v_j - p$ to rider $j$ for a completed trip at (incremental to base) price $p$.

We assume the platform has complete information about supply and demand over the planning horizon (driver and rider types, including driver entry during the planning horizon). Thus, the challenge addressed in this paper is one of promoting desirable behavior in the absence of time-extended contracts, and not one of information asymmetry. We assume drivers have the same information, and that this is common knowledge amongst drivers.[17] Unless otherwise noted, we assume properties (S1), (S2), and complete, symmetric information throughout the paper, and discuss the effect of relaxing these assumptions in Section 4.3.

At each time $t$, a driver is *en route* if she started her last trip from $a$ to $b$ at time $t'$ (with or without a rider), and $t < t' + \delta(a, b)$. A driver is *available* if she has entered the platform ($t \geq \underline{\tau}_i$) and is not *en route*. A driver who is available at time $t$ and location $a$ is able to complete a pick-up at this location and time. We allow a driver to drop-off a rider and pick-up another rider in the same location at the same time point.

A *path* is a sequence of tuples $(a, b, t)$, representing the trips taken by a driver over the planning horizon. A *feasible path* for driver $i$ starts at $(\ell_i, \underline{\tau}_i)$, with the starting time and location of each successive trip equal to the ending time and location of the previous trip. Let $\mathcal{Z}_i$ denote the set of

[14] We can also allow the distance between a pair of locations to change over time, modeling the changes in traffic conditions, i.e. a trip from $a$ to $b$ starting at time $t$ ends at time $t + \delta(a, b, t)$. This does not affect the results presented in this paper, and we keep $\delta(a, b)$ for simplicity of notation.

[15] With the triangle inequality, i.e. $\delta(a, c) \leq \delta(a, b) + \delta(b, c)$ for all $a, b, c \in \mathcal{L}$, riders would not have incentives in the STP mechanism to break a long trip into several shorter trips in order to get a lower price. See Section 4.3 for discussions on rider incentives.

[16] The base payment can be thought to cover, for example, the unit costs to drivers of driving and the opportunity cost for other outside employment options. We suppose the base payment is collected and distributed to drivers in some trip-based manner (e.g. through per mile and per minute rates, and may be non-zero even when the prices determined by the STP mechanism is zero).

[17] More generally, it is sufficient that it be common knowledge amongst drivers that the platform has the correct information.

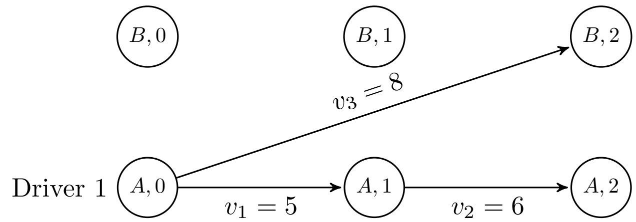



Figure 1: The economy in Example 1, with two locations $A$, $B$, two time periods and three riders.

all feasible paths of driver $i$, with $Z_{i,k} \in \mathcal{Z}_i$ to denote the $k^{\text{th}}$ feasible path. Denote $(a, b, t) \in Z_{i,k}$ if path $Z_{i,k}$ includes (or *covers*) trip $(a, b, t)$. Driver $i$ that takes the path $Z_{i,k}$ is able to pick up rider $j$ if $(o_j, d_j, \tau_j) \in Z_{i,k}$, however, a path specifies only the movement in space and time, and does not specify whether a/which rider is picked up for each of the trips on the path.

Let an *action path* for driver $i$ be a sequence of tuples, each of them can either be of the form $(a, b, t)$, representing a relocation trip from $a$ to $b$ at time $t$ without a rider, or be of the form $(a, b, t, j)$, in which case the driver sends rider $j$ from $a$ to $b$ at time $t$ (thus requiring $(a, b, t) = (o_j, d_j, \tau_j)$). Let $\tilde{\mathcal{Z}}_i$ be the set of all feasible action paths of driver $i$ (the feasibility of an action path is similar to that of a path). For an action path $\tilde{z}_i \in \tilde{\mathcal{Z}}_i$, denote $(a, b, t) \in \tilde{z}_i$ or $(a, b, t, j) \in \tilde{z}_i$ if the action path includes a relocation or rider trip from $a$ to $b$ at time $t$.

**Example 1.** The planning horizon is $T = 2$ and there are two locations $\mathcal{L} = \{A, B\}$, with distance $\delta(A, A) = \delta(B, B) = 1$ and $\delta(A, B) = \delta(B, A) = 2$. See Figure 1. There is one driver, who enters at time $\underline{\tau}_1 = 0$ at location $\ell_1 = A$ and leaves at time $\bar{\tau}_1 = 2$. There are three riders:

- Rider 1: $o_1 = A$, $d_1 = A$, $\tau_1 = 0$, $v_1 = 5$,
- Rider 2: $o_2 = A$, $d_2 = A$, $\tau_2 = 1$, $v_1 = 6$,
- Rider 3: $o_3 = A$, $d_3 = B$, $\tau_3 = 0$, $v_3 = 8$.

There are two feasible paths for driver 1: $Z_{1,1} = ((A, A, 0),\ (A, A, 1))$ and $Z_{1,2} = ((A, B, 0))$. Each of "picking up riders 1 and 2" and "staying static at location $A$ for two periods" are consistent with path $Z_{1,1}$. Path $((A, A, 0),\ (A, B, 1))$ is infeasible, since the last trip ends later than the driver's leaving time. Similarly, paths $((A, B, 0),\ (B, B, 1))$ and $((A, A, 0), (B, B, 1))$ are infeasible.

There are a total of six feasible actions paths of rider 1: $((A, B, 0))$— relocating from $A$ to $B$ at time 0, $((A, B, 0, 3))$— sending rider 3 from $A$ to $B$ at time 0, and similarly, $((A, A, 0), (A, A, 1))$, $((A, A, 0, 1), (A, A, 1))$, $((A, A, 0), (A, A, 1, 2))$, and $((A, A, 0, 1), (A, A, 1, 2))$. $\square$

We now provide an informal definition of a *ridesharing mechanism*. At each time point $t \in [T-1]$, given the history of trips, current driver locations, driver availability status, and information about future driver supply and rider demand for trips, a ridesharing mechanism:

1. Determines for each rider with trip start time $t$, whether a driver will be dispatched to pick her up, and if so, the price of her trip.

2. Dispatches available drivers to pick up riders or to relocate, and the payments offered to drivers for accepting the dispatches.

3. Each driver decides whether to accept the dispatch, or deviate and either stay in the same location or relocate. The mechanism collects and makes payments based on driver actions.

Any undispatched, available driver makes their own choices of actions, and we assume that any driver already *en route* will continue their current trip. A driver's payment in a period in which the driver declines a dispatch is zero, so that drivers are not charged penalties for deviation.

As a baseline, we define the following mechanism:

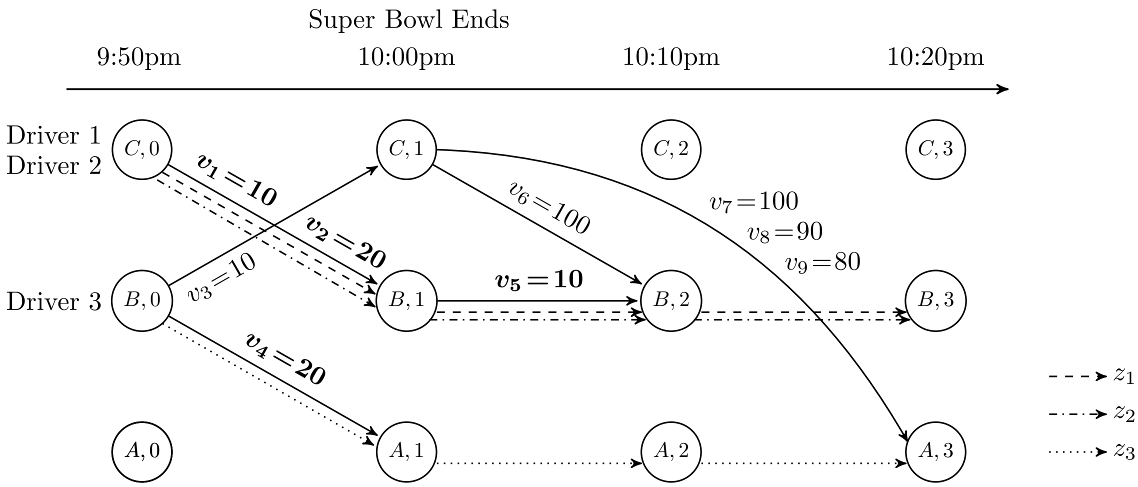



Figure 2: A Super Bowl game: time 0 plan under the myopic pricing mechanism.

**Definition 1** (Myopic pricing mechanism)**.** At each time point $t \in [T]$, for each location $a \in \mathcal{L}$, the *myopic pricing mechanism* dispatches available drivers at $(a, t)$ to riders requesting rides from $(a, t)$ in decreasing order of riders' values, and sets a market clearing price $p_{a,t}$ that is offered to all dispatched drivers (and will be collected from riders).

The market clearing prices may not be unique, and a fully defined myopic mechanism must provide a rule for picking a particular set of clearing prices. This mechanism has anonymous, origin-based pricing, and is very simple in ignoring the need for smooth pricing, or future supply and demand.

**Example 1** (Continued)**.** In Example 1, the myopic pricing mechanism dispatches driver 1 to pick up rider 3 at time 0 since $v_3 > v_1$. Any price $p_{A,0} \in [5,\ 8]$ clears the market. At time 1, no driver is able to pick up rider 2, thus any $p_{A,1} \geq 6$ clears the market. The welfare, that is the total value of riders that are picked up, is $v_3 = 8$, whereas it is possible to achieve a total welfare of $v_1 + v_2 = 11$ by dispatching driver 1 to pick rider 1 and then rider 2. $\square$

**Example 2** (Super Bowl example)**.** Consider the economy illustrated in Figure 2, modeling the end of a sports event, with three time periods and three locations. Time $t = 0$ is 9:50pm, and 10 minutes before the Super Bowl ends, and each time period is 10 minutes. There are three locations $A$, $B$ and $C$ with symmetric distances $\delta(A, A) = \delta(B, B) = \delta(C, C) = \delta(A, B) = \delta(B, A) = \delta(B, C) = \delta(C, B) = 1$ and $\delta(A, C) = \delta(C, A) = 2$. Drivers 1 and 2 enter at location $C$ at time 0, while driver 3 enters at location $B$ at time 0. At time 1, many riders with very high value show up at location $C$, where the game takes place. Riders' trips and their willingness to pay are as shown in the figure.

Under the myopic pricing mechanism, at time 0, drivers 1 and 2 are dispatched to pick up riders 1 and 2 respectively, and driver 3 is dispatched to pick up rider 4. At time 1, one of the two drivers at location $B$ picks up rider 5, and the total social welfare would be $v_1 + v_2 + v_4 + v_5 = 60$. The set of market clearing prices for the trip $(C, B, 0)$ is $p_{C,B,0} \in [0,\ 10]$, and the price for $(B, B, 1)$ is $p_{B,B,1} = 0$ since there is excessive supply. The highest possible total payment to driver 1 under any myopic pricing mechanism would be 10. At time 1, since no driver is able to pick up the four riders at location $C$, the lowest market clearing prices are $p_{C,B,1} = p_{C,A,1} \geq 100$. Suppose driver 1 deviates and stays in location $C$ until time 1. The mechanism would then dispatch driver 1 to pick up rider 6, and driver 1 would be paid the new market clearing price of 100. This is a useful deviation. Moreover, by strategizing, driver 1 improves the social welfare by making use of information about the future demand and supply. □

# 3 A Static CE Mechanism

In this section, we formulate the optimal planning problem and define competitive equilibrium (CE) prices. We also introduce the *static CE mechanism*, which computes an optimal plan at the start of the planning horizon and does not replan in the future. This makes the static CE mechanism fragile, for example it is not welfare optimal or envy-free forward from a period in which some driver has deviated from the suggested plan. The STP mechanism resolves this concern, combining driver-pessimal CE prices with replanning to provide robustness forward from any history. At the same time, this promise to replan will also bring new strategic concerns into the analysis of the STP mechanism.

## 3.1 Plans

A *plan* describes the paths taken by all drivers until the end of the planning horizon, rider pick-ups, as well as payments for riders and drivers for each trip associated with these paths.

Formally, a plan is the 4-tuple $(x, \tilde{z}, q, \pi)$, where: $x$ is the indicator of rider pick-ups, where for all $j \in \mathcal{R}$, $x_j = 1$ if rider $j$ is picked-up according to the plan, and $x_j = 0$ otherwise; $\tilde{z}$ is a vector of action paths, where $\tilde{z}_i \in \tilde{\mathcal{Z}}_i$ is the dispatched action path for driver $i$; $q_j$ denotes the payment made by rider $j$, $\pi_{i,t}$ denotes the payment made to driver $i$ at time $t$, and let $\pi_i \triangleq \sum_{t=0}^{T} \pi_{i,t}$ denote the total payment to driver $i$.

A plan $(x, \tilde{z}, q, \pi)$ is *feasible* if for each rider $j \in \mathcal{R}$, $x_j = \sum_{i \in \mathcal{D}} \mathbb{1}\{(o_j, d_j, \tau_j, j) \in \tilde{z}_i\} \in \{0,\ 1\}$, where $\mathbb{1}\{\cdot\}$ is the indicator function. Unless stated otherwise, when we mention a plan in the rest of the paper, it is assumed to be feasible. For the *budget balance* (BB) of a plan, we need:

$$\sum_{j \in \mathcal{R}} q_j \geq \sum_{i \in \mathcal{D}} \pi_i, \tag{1}$$

with strict budget balance if (1) holds with equality. *Individually rational* (IR) for riders requires

$$x_j v_j \geq q_j,\ \forall j \in \mathcal{R}. \tag{2}$$

A plan is *envy-free for drivers* if any pair of drivers entering the platform at the same location and time get paid the same amount:

$$\pi_i = \pi_{i'} \text{ for all } i,\ i' \in \mathcal{D} \text{ s.t. } \underline{\tau}_i = \underline{\tau}_{i'},\ \text{ and } \ell_i = \ell_{i'} \tag{3}$$

A plan is *envy-free for riders* if no rider strictly prefers the outcome of another rider requesting the same trip, that is

$$x_j v_j - q_j \geq x_{j'} v_j - q_{j'} \text{ for all } j,\ j' \in \mathcal{R} \text{ s.t. } o_j = o_{j'},\ d_j = d_{j'},\ \text{and } \tau_j = \tau_{j'}. \tag{4}$$

**Definition 2** (Anonymous trip prices)**.** A plan $(x, \tilde{z}, q, \pi)$ uses *anonymous trip prices* if there exist $p = \{p_{a,b,t}\}_{(a,b,t) \in \mathcal{T}}$ such that for all $(a, b, t) \in \mathcal{T}$, we have:

(i) all riders taking the same $(a, b, t)$ trip are charged the same payment $p_{a,b,t}$, and there is no payment by riders who are not picked up, and

(ii) all drivers that are dispatched on a rider trip from $a$ to $b$ at time $t$ are paid the same amount $p_{a,b,t}$ for the trip at time $t$, and there is no payment to drivers for relocation, or at times a driver is en route finishing up a previously dispatched trips.

Given dispatches $(x, \tilde{z})$ and anonymous trips prices $p$, all payments are fully determined: the total payment to driver $i$ is $\pi_i = \sum_{j \in \mathcal{R}} \mathbb{1}\{(o_j, d_j, \tau_j, j) \in \tilde{z}_i\} p_{o_j,d_j,\tau_j}$ and the payment made by rider $j$ is $q_j = x_j p_{o_j,d_j,\tau_j}$. For this reason, we will represent plans with anonymous trip prices as $(x, \tilde{z}, p)$. By construction, plans with anonymous trip prices are strictly budget balanced.

**Definition 3** (Competitive Equilibrium)**.** A plan with anonymous trip prices $(x, \tilde{z}, p)$ forms a *competitive equilibrium* (CE) if:

(i) (rider best response) all riders $j \in \mathcal{R}$ that can afford the ride are picked up, i.e. $v_j > p_{o_j,d_j,\tau_j} \Rightarrow x_j = 1$, and all riders that are picked up can afford the price: $x_j = 1 \Rightarrow v_j \geq p_{o_j,d_j,\tau_j}$,

(ii) (driver best response) $\forall i \in \mathcal{D}$, $\pi_i = \max_{z_i \in \mathcal{Z}_i} \left\{ \sum_{(a,b,t) \in z_i} \max\{p_{a,b,t}, 0\} \right\}$, i.e. each driver gets paid the highest achievable amount given prices and the set of feasible paths.

Given any set of anonymous trip prices $p$, let anonymous trip prices $p^+$ be defined as $p^+_{a,b,t} \triangleq \max\{p_{a,b,t},\ 0\}$ for each $(a,b,t) \in \mathcal{T}$.

**Lemma 1.** Given any CE plan $(x, \tilde{z}, p)$, the plan with anonymous prices $(x, \tilde{z}, p^+)$ also forms a CE, and has the same driver and rider payments as those under the original CE plan $(x, \tilde{z}, p)$.

The lemma implies that when studying the set of possible rider and driver payments among all CE outcomes, it is without loss to consider only anonymous trip prices that are non-negative. We leave the full proof to Appendix B.1. Prices must be non-negative for any trip that is requested by any rider, thus changing prices from $p$ to $p^+$ does not affect the payments for any rider or driver, or the best response on the riders side. The driver best response property also continues to hold, since $\max\{p_{a,b,t}, 0\} = \max\{p^+_{a,b,t}, 0\}$ for all $(a,b,t) \in \mathcal{T}$.

Given any mechanism, complete information about supply and demand, and assuming that all drivers follow the dispatches of the mechanism at all times, the assignments of all riders, action paths taken by all drivers, and the corresponding payment schedule through the planning horizon can be computed at time 0, if all available drivers are dispatched at all times. We call this outcome the "time 0 plan" under the given mechanism.

**Example 1** (Continued)**.** For the economy in Example 1, and for the myopic pricing mechanism with lowest market clearing prices, the time 0 plan has driver 1 taking the action path $((A, B, 0, 3))$, with anonymous trip prices $p_{A,A,0} = p_{A,B,0} = 5$ and $p_{A,A,1} \geq 6$. This time 0 plan does not form a CE, since the driver is paid $\pi_1 = 5$, whereas the total payment for the other feasible path of the driver $((A, A, 0),\ (A, A, 1))$ is $p_{A,A,0} + p_{A,A,1} = 11 > 5$. Consider an alternative plan, where the anonymous trips prices are $p_{A,A,0} = p_{A,A,1} = 4$ and $p_{A,B,0} = 8$, and the driver takes the action path $((A, A, 0, 1),\ (A, A, 1, 2))$ and picks up riders 1 and 2. In this case, the time 0 plan forms a CE. □

## 3.2 Optimal Plans and CE Prices

The welfare-optimal planning problem can be formulated as an integer linear program (ILP) that determines optimal rider pick-ups and driver paths, followed by an assignment of riders to drivers whose paths cover the rider trips. Let $x_j$ be the indicator that rider $j \in \mathcal{R}$ is picked up, and $y_{i,k}$ be the indicator that driver $i$ takes $Z_{i,k}$, her $k^{\text{th}}$ feasible path in $\mathcal{Z}_i$. We have:

$$\max_{x,y} \quad \sum_{j \in \mathcal{R}} x_j v_j \tag{5}$$

$$\text{s.t.} \quad \sum_{j \in \mathcal{R}} x_j \mathbb{1}\{(o_j, d_j, \tau_j) = (a, b, t)\} \le \sum_{i \in \mathcal{D}} \sum_{k=1}^{|\mathcal{Z}_i|} y_{i,k} \mathbb{1}\{(a, b, t) \in Z_{i,k}\}, \qquad \forall (a, b, t) \in \mathcal{T} \tag{6}$$

$$\sum_{k=1}^{|\mathcal{Z}_i|} y_{i,k} \le 1, \qquad \forall i \in \mathcal{D} \tag{7}$$

$$x_j \in \{0, 1\}, \qquad \forall j \in \mathcal{R} \tag{8}$$

$$y_{i,k} \in \{0, 1\}, \qquad \forall i \in \mathcal{D},\ k = 1, \dots, |\mathcal{Z}_i| \tag{9}$$

The feasibility constraint (6) requires that for all trips $(a, b, t) \in \mathcal{T}$, the number of riders who request this trip and are picked up is no more than the total number of drivers whose paths cover this trip. (7) requires that each driver takes at most one path. Once pick-ups $x$ and paths $y$ are computed, (6) guarantees that each rider with $x_j = 1$ can be assigned to a driver.

Relaxing the integrality constraints on variables $x$ and $y$, we obtain the following linear program (LP) relaxation of the ILP:

$$\max_{x,y} \quad \sum_{j \in \mathcal{R}} x_j v_j \tag{10}$$

$$\text{s.t.} \quad \sum_{j \in \mathcal{R}} x_j \mathbb{1}\{(o_j, d_j, \tau_j) = (a, b, t)\} \le \sum_{i \in \mathcal{D}} \sum_{k=1}^{|\mathcal{Z}_i|} y_{i,k} \mathbb{1}\{(a, b, t) \in Z_{i,k}\}, \qquad \forall (a, b, t) \in \mathcal{T} \tag{11}$$

$$\sum_{k=1}^{|\mathcal{Z}_i|} y_{i,k} \le 1, \qquad \forall i \in \mathcal{D} \tag{12}$$

$$x_j \le 1, \qquad \forall j \in \mathcal{R} \tag{13}$$

$$x_j \ge 0, \qquad \forall j \in \mathcal{R} \tag{14}$$

$$y_{i,k} \ge 0, \qquad \forall i \in \mathcal{D},\ k = 1, \dots, |\mathcal{Z}_i| \tag{15}$$

We refer to (10) as the primal LP. The constraint $y_{i,k} \le 1$, that each path is taken by each driver at most once is guaranteed by imposing (12), and is omitted.

**Lemma 2** (Integrality)**.** There exists an integer optimal solution to the linear program (10).

We leave the proof to Appendix B.2, showing there a correspondence to a min cost flow (MCF) problem (that has integral optimal solutions), where drivers flow through a network with vertices corresponding to (location, time) pairs, edges corresponding to trips, and with edge costs equal to the negation of riders' values. The reduction to MCF can also be used to solve the LP efficiently.

Let $p_{a,b,t}$, $\pi_i$ and $u_j$ denote the dual variables corresponding to the primal constraints (11), (12) and (13), respectively. The dual LP of (10) is:

$$\min \sum_{i\in\mathcal{D}} \pi_i + \sum_{j\in\mathcal{R}} u_j \tag{16}$$

$$\text{s.t. } \pi_i \geq \sum_{(a,b,t)\in Z_{i,k}} p_{a,b,t}, \qquad \forall k=1,\dots,|\mathcal{Z}_i|,\ \forall i\in\mathcal{D} \tag{17}$$

$$u_j \geq v_j - p_{o_j,d_j,\tau_j}, \qquad \forall j\in\mathcal{R} \tag{18}$$

$$p_{a,b,t} \geq 0, \qquad \forall (a,b,t)\in\mathcal{T} \tag{19}$$

$$\pi_i \geq 0, \qquad \forall i\in\mathcal{D} \tag{20}$$

$$u_j \geq 0, \qquad \forall j\in\mathcal{R} \tag{21}$$

**Lemma 3** (Welfare Theorem)**.** A dispatching $(x,\tilde{z})$ is welfare-optimal if and only if there exists anonymous trip prices $p$ s.t. the plan $(x,\tilde{z},p)$ forms a competitive equilibrium. Such optimal CE plans always exist, are efficient to compute, and are individually rational for riders, strictly budget balanced, and envy-free for both riders and drivers.

Given optimal primal and dual solutions, we show that the dual variables $\pi$ and $u$ can be interpreted as the payments to drivers and utilities of riders, when the anonymous trip prices are given by $p$. We then make use of Lemma 1, and the standard observations about complementary slackness conditions and their connection with competitive equilibria [30, 7]. By integrality, optimal CE plans always exist, and can be efficiently computed by solving the primal and dual LPs of the MCF problem. See Appendix B.3 for the proof of the lemma.

For any two driver payment profiles $\pi=(\pi_1,\dots,\pi_m)$ and $\pi'=(\pi_1',\dots,\pi_m')$ that correspond to optimal CE plans, let the join $\bar{\pi}=\pi\vee\pi'$ and the meet $\underline{\pi}=\pi\wedge\pi'$ be defined as $\bar{\pi}_i=\max\{\pi_i,\pi_i'\}$ and $\underline{\pi}_i=\min\{\pi_i,\pi_i'\}$ for all $i\in\mathcal{D}$. The following lemma shows that drivers' total payments among all CE outcomes form a lattice, meaning that there exist CE plans where the driver payment profiles are given by $\bar{\pi}$ or $\underline{\pi}$. The lemma also shows a connection between the top/bottom of the lattice and the welfare differences from losing/replicating a driver, which plays an important role in establishing the incentive properties of the STP mechanism. Denote $W(\mathcal{D})$ as the highest welfare achievable by a set of drivers $\mathcal{D}$. For each driver $i\in\mathcal{D}$, define the *social welfare gain from replicating driver* $i$, and the *social welfare loss from eliminating driver* $i$, as:

$$\Phi_{\ell_i,\underline{\tau}_i} \triangleq W(\mathcal{D}\cup\{(\underline{\tau}_i,\bar{\tau}_i,\ell_i)\}) - W(\mathcal{D}), \tag{22}$$

$$\Psi_{\ell_i,\underline{\tau}_i} \triangleq W(\mathcal{D}) - W(\mathcal{D}\backslash\{(\underline{\tau}_i,\bar{\tau}_i,\ell_i)\}), \tag{23}$$

A *driver-optimal plan* has a driver payment profile at the top of the lattice, and a *driver-pessimal plan* has a payment profile at the bottom of the lattice.

**Lemma 4** (Lattice Structure)**.** Drivers' total payments $\pi$ among all CE outcomes form a lattice. Moreover, for each driver $i\in\mathcal{D}$, $\Phi_{\ell_i,\underline{\tau}_i}$ and $\Psi_{\ell_i,\underline{\tau}_i}$ are equal to the total payments to driver $i$ in the driver-pessimal and driver-optimal CE plans, respectively.

We leave the proof of Lemma 4 to Appendix B.4.The lattice structure follows from the correspondences between driver payments, optimal solutions to dual LP (16), and optimal solutions to the dual of the flow LP, and the fact that optimal duals of MCF form a lattice. Standard arguments on shortest paths in the residual graph [2], and the connection between optimal dual solutions and subgradients (w.r.t. flow boundary conditions), then imply the correspondence between welfare gains/losses and driver pessimal/optimal payments.

Unlike the classical unit-demand assignment problem, where the prices of items and the utilities of buyers both form lattices [33], it is drivers' payments, and not trip prices or rider utilities that have a lattice structure. This is because although the driver-pessimal payment is unique, the trip prices under all driver-pessimal plans need not be unique. See Section 4.3 for further a development.

### 3.3 The Static CE Mechanism

A *static mechanism* is one that announces a plan $(x, \tilde{z}, q, \pi)$ at time $t = 0$, without updating the plan after any driver deviation. Rather, each driver can choose to take any feasible path, but can only pick up riders that are dispatched to her (and is only paid for the subset of these rider trips that are completed). Riders pay according to prices $q$ in the event they are picked up by a driver.

**Definition 4** (Static CE Mechanism)**.** A static CE mechanism announces an optimal CE plan $(x, \tilde{z}, p)$ at the beginning of the planning horizon. Each driver $i \in \mathcal{D}$ then decides on the actual action path $\tilde{z}_i'$ that she takes, and gets paid $\hat{\pi}_i = \sum_{j \in \mathcal{R}} p_{o_j, d_j, \tau_j} \mathbb{1}\{(o_j, d_j, \tau_j, j) \in \tilde{z}_i,\ (o_j, d_j, \tau_j, j) \in \tilde{z}_i'\}$. Each rider $j \in \mathcal{R}$ pays $q_j$ only if she is picked up.

A static CE mechanism can be defined for any set of CE prices. Since a driver can only pick up and get paid for rider trips that she was originally dispatched given $\tilde{z}_i$, then no other path gives her a higher total payment, and it is a dominant strategy for each driver to follow $\tilde{z}_i$.

**Theorem 1.** *A static CE mechanism implements an optimal CE plan in dominant strategy.*

In addition, if all riders and drivers follow the plan, the outcome under a static CE mechanism is strictly budget balanced and envy-free. The CE property also ensures that every rider that is picked up is happy to take the trip at the offered price, and that no rider who is not picked up has positive utility for the trip.[18]

The optimal static mechanism enjoys many good properties, but has a decisive design flaw—it is fragile to driver deviations since it does not react by replanning. Deviations could occur for many reasons: mistakes, unexpected contingencies, unexpected traffic, or unmodeled idiosyncratic preferences. We show by revisiting the Super Bowl example that once a driver has deviated, the resulting outcome in the subsequent periods may no longer be welfare-optimal or envy-free.

**Example 2** (Continued)**.** For the Super Bowl game scenario, the static CE mechanism with a driver-pessimal plan adopts the plan illustrated in Figure 3. The anonymous trip prices are shown in italics, below the edges corresponding to the trips. In this plan, all drivers stay at $C$ or re-position to location $C$ at time 0, pick up riders with high values, and achieve the optimal welfare of 300. The outcome forms a CE, that there is no other path with a higher total payment for any driver. All riders are happy with their dispatched trips given the prices, and there is no driver or rider envy.

Suppose now that driver 3 did not follow the plan at time 0 to pick-up rider 3 going from $B$ to $C$, but stayed in location $B$ until time 1. The effect of this deviation and not updating the plan is that driver 3 is no longer able to pick up rider 6, who strictly prefers to be picked up given the original price of $p_{B,B,2} = 80$. One of the drivers 1 and 2 who was supposed to pick up rider 8 with value 90 is actually able to pick up rider 6, who has a higher value. Moreover, driver 3 is now able to pick up rider 5, however, she wouldn't be dispatched to do so. □

[18]Still, Example 8 in Appendix C.3 shows that truthful reporting of a rider's value need not be a dominant strategy (and this can be the case whichever CE prices are selected).

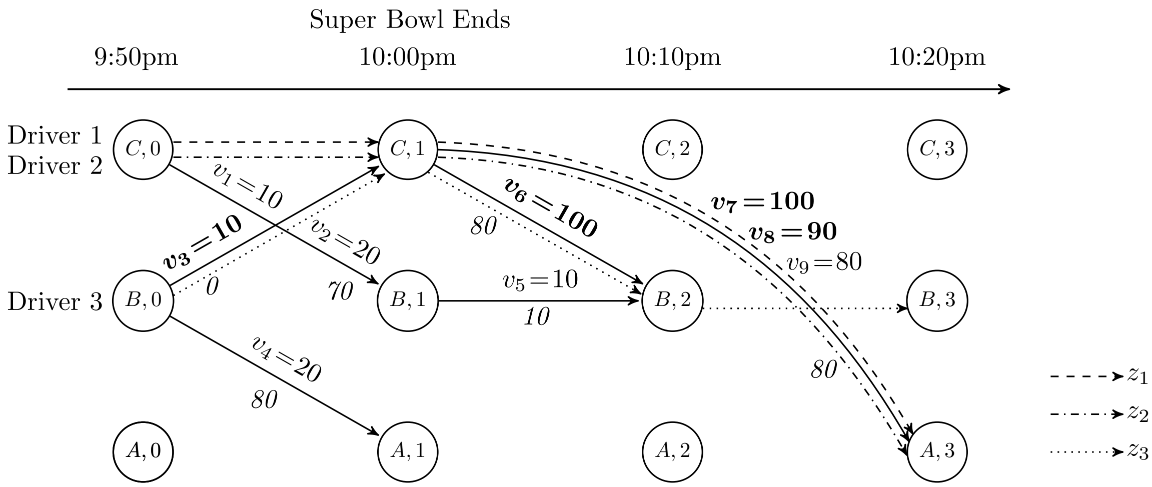


Figure 3: The Super Bowl example: the driver pessimal competitive equilibrium plan.

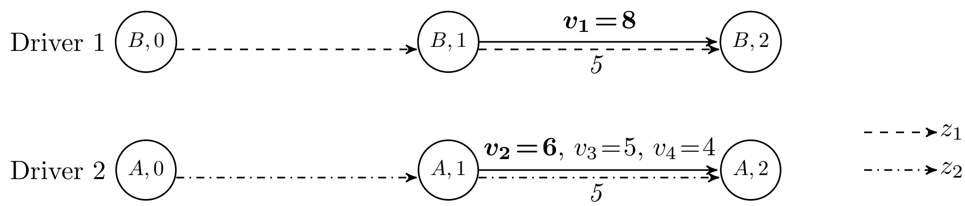


Figure 4: The economy in Example 3 and the driver pessimal CE plan computed at time 0.

One may think of a naive fix, simply repeating the computation of the plan of a static CE mechanism at all times. The following example shows that the mechanism that computes a driver pessimal plan at all times fails to be incentive compatible. Similarly, we show that the mechanism that repeatedly recomputes a *driver-optimal* plan is not envy-free, and can have incentive issues. See Example 10 in Appendix C.

**Example 3.** Consider the economy as shown in Figure 4, where there are two locations $\mathcal{L} = \{A, B\}$ with distances $\delta(A, A) = \delta(A, B) = \delta(B, A) = \delta(B, B) = 1$. In the driver-pessimal plan computed at time 0 as shown in the figure, the anonymous trip prices are $p_{B,B,1} = p_{A,A,1} = 5$. Assume that both drivers 1 and 2 follow the plan at time 0, and reach $(B, 1)$ and $(A, 1)$ respectively. If the mechanism re-computes the plan at time 1, the new driver-pessimal plan would set a new price of 0 for the trip $(B, B, 1)$— the updated lowest market-clearing price for the trip. Therefore, if driver 1 follows the mechanism at all times, her total payment would actually be 0. Now consider the scenario where driver 2 follows the mechanism at time 0, but driver 1 deviates and relocates to $A$, so that both drivers are at location $A$ at time 1. At time 1, when the mechanism recomputes a driver-pessimal plan, both drivers would take the trip $(A, A, 1)$ and pick up riders 2 and 3 respectively. The updated price for the trip $(A, A, 1)$ would be 4, and this is a useful deviation for driver 1. $\square$

The challenge is to achieve robustness to deviations, but at the same time handle the new strategic considerations that can occur as a result of drivers being able to trigger re-planning through deviations.

# 4 The Spatial-Temporal Pricing Mechanism

The Spatio-Temporal Pricing (STP) mechanism computes a driver-pessimal CE plan at the beginning of the planning horizon, and recomputes a driver-pessimal plan upon any deviation. This leads to our main result: the STP mechanism achieves subgame-perfect incentive compatibility without the use of time-extended contracts, and is welfare-optimal, individually rational, budget balanced, and envy-free from any history onward.

## 4.1 A Dynamic Mechanism

We first formally define a dynamic mechanism, that can use the history of actions to update the plan forward from the current state.

Let $s_t = (s_{1,t}, s_{2,t}, \ldots, s_{m,t})$ denote the *state* of the ridesharing platform at time $t$, where each $s_{i,t}$ describes the state of driver $i \in \mathcal{D}$. If driver $i$ has entered the platform and is available at time $t$ at location $a \in \mathcal{L}$, denote $s_{i,t} = (a, t)$. Otherwise, if driver $i$ is *en route*, finishing the trip from $a$ to $b$ that she started at time $t' < t$ s.t. $t' + \delta(a, b) > t$, denote $s_{i,t} = (a, b, t')$ if she is relocating, or $s_{i,t} = (a, b, t', j)$ if she is taking a rider $j$ from $a$ to $b$ at time $t'$. For drivers that have not yet entered, i.e. $\underline{\tau}_i > t$, we write $s_{i,t} = (\ell_i, \underline{\tau}_i)$. The initial state of the platform is $s_0 = ((\ell_1, \underline{\tau}_1), \ldots, (\ell_m, \underline{\tau}_m))$.

At each time $t$, each driver $i$ takes an *action* $\alpha_{i,t}$. An available driver $i$ with $s_{i,t} = (a, t)$ may relocate to any location $b$ within reach by the end of the planning horizon (i.e. $b \in \mathcal{L}$ s.t. $t + \delta(a, b) \leq T$), which we denote $\alpha_{i,t} = (a, b, t)$. She may also pick up a rider $j \in \mathcal{R}$ with $\tau_j = t$ and $o_j = a$, in which case we write $\alpha_{i,t} = (a, d_j, t, j)$. For a driver $i$ that is *en route* at time $t$, (i.e. $s_{i,t} = (a, b, t')$ or $s_{i,t} = (a, b, t', j)$ for some $t'$ s.t. $t' + \delta(a, b) > t$), $\alpha_{i,t} = s_{i,t}$— the only available action is to finish the current trip. For driver $i$ that has not yet entered, denote $\alpha_{i,t} = s_{i,t} = (\underline{\tau}_i, \ell_i)$.

The action $\alpha_{i,t}$ taken by driver $i$ at time $t$ determines her state $s_{i,t+1}$ at time $t + 1$:

- (will complete trips at $t+1$) if $\alpha_{i,t} = (a, b, t')$ or $\alpha_{i,t} = (a, b, t', j)$ s.t. $t' + \delta(a, b) = t+1$, then $s_{i,t+1} = (b, t+1)$, meaning these drivers will become available at time $t+1$ at the destination of their trips,[19]
- (still *en route*) if $\alpha_{i,t} = (a, b, t')$ or $\alpha_{i,t} = (a, b, t', j)$ s.t. $t' + \delta(a, b) > t + 1$, then $s_{i,t+1} = \alpha_{i,t}$,
- (not yet entered) if $i \in \mathcal{D}$ s.t. $\alpha_{i,t} = (\underline{\tau}_i, \ell_i)$, then $s_{i,t+1} = (\underline{\tau}_i, \ell_i)$.

Let $\alpha_t = (\alpha_{1,t}, \alpha_{2,t}, \ldots, \alpha_{m,t})$ be the *action profile* of all drivers at time $t$, and let *history* $h_t \triangleq (s_0, \alpha_0, s_1, \alpha_1, \ldots, s_{t-1}, \alpha_{t-1}, s_t)$, with $h_0 = (s_0)$. Finally, let $\mathcal{D}_t(h_t) = \{i \in \mathcal{D} \mid s_{i,t} = (a, t) \text{ for some } a \in \mathcal{L}\}$ be the set of drivers available at time $t$.

**Definition 5** (Dynamic Ridesharing Mechanism)**.** A dynamic ridesharing mechanism is defined by its *dispatch rule* $\alpha^*$, *driver payment rule* $\pi^*$ and *rider payment rule* $q^*$. At each time $t$, given history $h_t$ and rider information $\mathcal{R}$, the mechanism:

- uses its dispatch rule $\alpha^*$ to determine for each of a subset of available drivers, a *dispatch action* $\alpha^*_{i,t}(h_t)$ to either pick up a rider or to relocate.
- uses its driver payment rule $\pi^*$ to determine, for each dispatched driver, a payment $\pi^*_{i,t}(h_t)$ in the event the driver takes the action ($\pi^*_{i,t}(h_t) = 0$ for available drivers that are not dispatched).

[19] Here we assume that a driver that declines the mechanism's dispatch and decide to relocate from $a$ to $b$ also does so in time $\delta(a, b)$. We can also handle drivers who move more slowly when deviating, just as long as the mechanism knows when and where the driver will become available again.

- dispatches each *en route* driver to keep driving (i.e. $\alpha^*_{i,t}(h_t) = s_{i,t}$), and does not make any payment to driver $i$ in this period: $\pi^*_{i,t}(h_t) = 0$.
- uses its rider payment rule $q^*$ to determine, for each rider who receives a dispatch at time $t$, the payment $q^*_j(h_t)$ in the event that the rider is picked up.

Each driver then decides on which action $\alpha_{i,t} \in \mathcal{A}_{i,t}(h_t)$ to take, where $\mathcal{A}_{i,t}(h_t)$ is the set of actions available to agent $i$ at time $t$ given history $h_t$. For an available driver at $(a,t)$ with dispatched action $\alpha^*_{i,t}(h_t)$, $\mathcal{A}_{i,t}(h_t) = \{\alpha^*_{i,t}(h_t)\} \cup \{(a,b,t) \mid b \in \mathcal{L} \text{ s.t. } t+\delta(a,b) \le T\}$, i.e. the driver can either take the dispatched action, or to relocate to any location; if an available driver at $(a,t)$ is not dispatched, $\alpha^*_{i,t}(h_t)$, $\mathcal{A}_{i,t}(h_t) = \{(a,b,t) \mid b \in \mathcal{L} \text{ s.t. } t+\delta(a,b) \le T\}$; for an *en route* driver or a driver that has not yet entered the platform, $\mathcal{A}_{i,t}(h_t) = \{s_{i,t}\}$. After observing the action profile $\alpha_t$, the mechanism pays each dispatched driver $\hat{\pi}_{i,t}(\alpha_{i,t}, h_t) = \pi^*_{i,t}(h_t)\mathbb{1}\{\alpha_{i,t} = \alpha^*_{i,t}\}$, and charges each rider $j \in \mathcal{R}$ with $\tau_j = t$ the amount $\hat{q}_j(\alpha_t) = q^*_j(h_t)\sum_{i\in\mathcal{D}_t}\mathbb{1}\{\alpha_{i,t} = (o_j, d_j, t, j)\}$.

A mechanism is *feasible* if (i) it is possible for each available driver to take the dispatched trip, i.e. $\forall t$, $\forall h_t$, $\forall i \in \mathcal{D}_t$, if $s_{i,t} = (a,t)$ for some $a \in \mathcal{L}$, $\alpha^*_{i,t}(h_t) \in \{(a,b,t) \mid b \in \mathcal{L},\ t+\delta(a,b) \le T\} \cup \{(o_j, d_j, \tau_j, j) \mid j \in \mathcal{R},\ \tau_j = t,\ o_j = a\}$, and (ii) no rider is picked-up more than once, i.e. $\forall t$, $\forall h_t$, $\forall j \in \mathcal{R}$ s.t. $\tau_j = t$, $\sum_{i\in\mathcal{D}_t}\mathbb{1}\{\alpha^*_{i,t}(h_t) = (o_j, d_j, \tau_j, j)\} \le 1$. From Definition 5, there is no payment to or from an *en route* or undispatched driver, a dispatched driver $i$ who deviated from $\alpha^*_{i,t}(h_t)$ at time $t$, or riders who are not picked up.

Let $\mathcal{H}_t$ be the set of all possible *histories* up to time $t$. A *strategy* $\sigma_i$ of driver $i$ defines for all times $t \in [T-1]$ and all histories $h_t \in \mathcal{H}_t$, the action she takes $\alpha_{i,t} = \sigma_i(h_t) \in \mathcal{A}_{i,t}(h_t)$. For a mechanism that always dispatches all available drivers, $\sigma^*_i$ denotes the *straightforward strategy* of always following the mechanism's dispatches at all times. Let $\sigma = (\sigma_1, \ldots, \sigma_m)$ be the *strategy profile*, with $\sigma_{-i} = (\sigma_1, \ldots, \sigma_{i-1}, \sigma_{i+1}, \ldots, \sigma_m)$. The strategy profile $\sigma$, together with the initial state $s_0$ and the rules of a mechanism, determine all actions and payments of all drivers through the planning horizon. Let $\sigma_i|_{h_t}$, $\sigma|_{h_t}$ and $\sigma_{-i}|_{h_t}$ denote the strategy profile from time $t$ and history $h_t$ onward for driver $i$, all drivers, and all drivers but $i$, respectively.

For each $i \in \mathcal{D}$, $\hat{\pi}_i(\sigma) = \sum_{t=0}^{T-1}\hat{\pi}_{i,t}(\sigma_i(h_t), h_t)$ denotes the total actual payments made to driver $i$, where drivers follow $\sigma$ and the history $h_t$ is induced by the initial state and strategy $\sigma$. For each $j \in \mathcal{R}$, let $\hat{x}_j(\sigma) \in \{0,1\}$ be the indicator that rider $j$ is picked-up given strategy $\sigma$, and let $\hat{q}_j(\sigma) = \hat{x}_j(\sigma)q^*_j(h_{\tau_j})$ be the payment made by rider $j$. Fixing driver and rider types, a ridesharing mechanism induces an extensive form game. At each time point $t$, each driver decides on an action $\alpha_{i,t} = \sigma_i(h_t) \in \mathcal{A}_{i,t}(h_t)$ to take based on strategy $\sigma_i$ and the history $h_t$, and receives payment $\hat{\pi}_{i,t}(\alpha_{i,t}, h_t)$. The total payment $\hat{\pi}_i(\sigma)$ to each driver is determined by the rules of the mechanism.

We define the following properties.

**Definition 6** (Budget Balance)**.** A ridesharing mechanism is *budget balanced* if for any set of riders and drivers, and any strategy profile $\sigma$ taken by the drivers, we have

$$\sum_{j\in\mathcal{R}}\hat{q}_j(\sigma) \ge \sum_{i\in\mathcal{D}}\hat{\pi}_i(\sigma). \tag{24}$$

**Definition 7** (Individual Rationality)**.** A ridesharing mechanism is *individually rational* for riders if for any set of riders and drivers, and any strategy profile $\sigma$ taken by the drivers,

$$\hat{x}_j(\sigma)v_j \ge \hat{q}_j(\sigma),\ \forall i \in \mathcal{R}. \tag{25}$$

**Definition 8** (Subgame-Perfect Incentive Compatibility)**.** A ridesharing mechanism that always dispatches all available drivers is *subgame-perfect incentive compatible* (SPIC) for drivers if given any set of riders and drivers, following the mechanism's dispatches at all times forms a subgame-perfect equilibrium (SPE) among the drivers, meaning for all $t \in [T-1]$, for any history $h_t \in \mathcal{H}_t$,

$$\sum_{t'=t}^{T-1} \hat{\pi}_{i,t'}(\sigma_i^*|_{h_t}, \sigma_{-i}^*|_{h_t}) \geq \sum_{t'=t}^{T-1} \hat{\pi}_{i,t'}(\sigma_i|_{h_t}, \sigma_{-i}^*|_{h_t}), \ \forall \sigma_i, \ \forall i \in \mathcal{D}. \tag{26}$$

A ridesharing mechanism is *dominant strategy incentive compatible (DSIC)* if for any driver, following the mechanism's dispatches at all time points that the driver is dispatched maximizes her total payment, regardless of the actions taken by the rest of the drivers.

**Definition 9** (Envy-freeness in SPE)**.** A ridesharing mechanism that always dispatches all available drivers is *envy-free in SPE for drivers* if for any set of riders and drivers, (i) the mechanism is SPIC for drivers, and (ii) for any time $t \in [T-1]$, for all history $h_t \in \mathcal{H}_t$, all drivers with the same state at time $t$ are paid the same total amount in the subsequent periods, assuming all drivers follow the mechanism's dispatches:

$$\sum_{t'=t}^{T-1} \hat{\pi}_{i,t'}(\sigma^*|_{h_t}) = \sum_{t'=t}^{T-1} \hat{\pi}_{i',t'}(\sigma^*|_{h_t}), \ \forall i, i' \in \mathcal{D} \text{ s.t. } s_{i,t} = s_{i',t}. \tag{27}$$

A ridesharing mechanism is *envy-free in SPE for riders* if (i) the mechanism is SPIC for drivers, and (ii) for all $j \in \mathcal{R}$, for all possible $h_{\tau_j} \in \mathcal{H}_{\tau_j}$, and all $j' \in \mathcal{R}$ s.t. $(o_j, d_j, \tau_j) = (o_{j'}, d_{j'}, \tau_{j'})$

$$\hat{x}_j(\sigma^*)v_j - \hat{q}_j(\sigma^*) \geq \hat{x}_{j'}(\sigma^*)v_j - \hat{q}_{j'}(\sigma^*). \tag{28}$$

Fix a ridesharing mechanism with dispatch rule $\alpha_{i,t}^*$ and payment rule $\pi_{i,t}^*$ where all available drivers are always dispatched. Recall that given complete information and the straightforward strategy $\sigma^*$, the outcome over the entire planning horizon can be computed at time 0, and is called the *time* 0 *plan* of the mechanism. If some driver deviated at time $t-1$ for some $t > 0$, the downward outcomes given the dispatching and payment rules, assuming all drivers follow $\sigma^*|_{h_t}$, can be thought of as an updated time $t$ plan.

For any time $t \in [T]$, given any state $s_t$ of the platform, let $E^{(t)}(s_t)$ represent the *time-shifted economy* starting at state $s_t$, with planning horizon $T^{(t)} = T - t$, the same set of locations $\mathcal{L}$ and distances $\delta$, and the remaining riders $\mathcal{R}^{(t)} = \{(o_j, d_j, \tau_j - t, v_j) \mid j \in \mathcal{R}, \tau_j \geq t\}$. For drivers, we have $\mathcal{D}^{(t)}(s_t) = \{(\underline{\tau}_i^{(t)}, \bar{\tau}_i^{(t)}, \ell_i^{(t)}) \mid i \in \mathcal{D}\}$, with types determined as follows:

- for available drivers $i \in \mathcal{D}$ s.t. $s_{i,t} = (a, t)$ for some $a \in \mathcal{L}$, $(\underline{\tau}_i^{(t)}, \bar{\tau}_i^{(t)}, \ell_i^{(t)}) = (0, \bar{\tau}_i - t, a)$,
- for *en route* drivers $i \in \mathcal{D}$ s.t. $s_{i,t} = (a, b, t')$ or $(a, b, t', j)$, $(\underline{\tau}_i^{(t)}, \bar{\tau}_i^{(t)}, \ell_i^{(t)}) = (t' + \delta(a, b) - t, \bar{\tau}_i - t, b)$, and
- for driver $i \in \mathcal{D}$ that had not entered the platform, $(\underline{\tau}_i^{(t)}, \bar{\tau}_i^{(t)}, \ell_i^{(t)}) = (\underline{\tau}_i - t, \bar{\tau}_i - t, \ell_i)$.

**Definition 10** (Temporal Consistency)**.** A ridesharing mechanism is *temporally consistent* if after deviation at time $t-1$, the updated plan is identical to that determined for economy $E^{(t)}(s_t)$.

Upon deviation(s) at time $t-1$, a temporally consistent mechanism computes its updated plan from time $t$ onward as if $t$ is the beginning of the planning horizon, thus the mechanism does not make use of time-extended contracts, including penalties for previous actions. In fact, a temporally inconsistent mechanism would be able to trivially align incentives. For example, a mechanism that replans based on the history could fire any driver who has deviated in the past, while keeping the plans for the rest of the economy unchanged. A mechanism can also threaten to "shut down" and not make any further dispatches or payments to the drivers if any of them had deviated.

## 4.2 The Spatio-Temporal Pricing Mechanism

We define the STP mechanism by providing a method to plan or re-plan, this implicitly defining the dispatch and payment rules. For each $a \in \mathcal{L}$ and $t \in [T]$, denote the welfare gain from an additional driver at $(a, t)$ as,

$$\Phi_{a,t} \triangleq W(\mathcal{D} \cup \{(t, T, a)\}) - W(\mathcal{D}), \tag{29}$$

where $(t, T, a)$ represents the type of this driver that stays until the end of the planning horizon.

**Definition 11** (Spatio-Temporal Pricing Mechanism)**.** The *spatio-temporal pricing (STP) mechanism* is a dynamic ridesharing mechanism that always dispatches all available drivers. Given economy $E^{(0)}$ at the beginning of the planning horizon, or economy $E^{(t)}(s_t)$ immediately after a deviation by one or more drivers, the mechanism completes the following planning step:

- *Dispatch rule:* To determine the dispatches ($\alpha^*$), compute an optimal solution $(x, y)$ to the ILP (5), and dispatch each driver $i$ to take the path $Z_{i,k}$ for $k$ s.t. $y_{i,k} = 1$ and pick up riders $j$ s.t. $x_j = 1$,
- *Payment rules:* To determine driver and rider payments ($\pi^*$ and $q^*$), for each $(a, b, t) \in \mathcal{T}$, set anonymous trip prices to be the difference in welfare gains: $p_{a,b,t} = \Phi_{a,t} - \Phi_{b,t+\delta(a,b)}$.
  - for each rider $j \in \mathcal{R}$, $q_j^* = p_{o_j,d_j,\tau_j} \sum_{i \in \mathcal{D}} \mathbb{1}\{\alpha^*_{i,\tau_j} = (o_j, d_j, \tau_j, j)\}$,
  - for each driver $i \in \mathcal{D}$, $\pi^*_{i,t} = \sum_{j \in \mathcal{R}, \tau_j = t} p_{o_j,d_j,t} \mathbb{1}\{\alpha^*_{i,\tau_j} = (o_j, d_j, t, j)\}$.

We now state the main result of the paper.

**Theorem 2.** *The spatio-temporal pricing mechanism is temporally consistent and subgame-perfect incentive compatible. It is also individually rational and strictly budget balanced for any action profile taken by the drivers, and is welfare optimal and envy-free in subgame-perfect equilibrium from any history onward.*

We provide the proof of Theorem 2 in Appendix B.5. We first observe that $\Phi_{a,T} = 0$ for all $a \in \mathcal{L}$ since an additional driver that enters at time $T$ cannot pick up any rider thus does not improve welfare. Any feasible path of each driver $i \in \mathcal{D}$ over the planning horizon starts at $(\ell_i, \underline{\tau}_i)$ and ends at $(a, T)$ for some $a \in \mathcal{L}$. By a telescoping sum, the total payment to driver $i$ under the planning rule of the STP mechanism is

$$\pi_i = \sum_{(a,b,t) \in z_i} p_{a,b,t} = \sum_{(a,b,t) \in z_i} \Phi_{a,t} - \Phi_{b,t+\delta(a,b)} = \Phi_{\ell_i, \underline{\tau}_i} - 0 = \Phi_{\ell_i, \underline{\tau}_i}, \tag{30}$$

and the welfare gain of the economy from replicating driver $i$. Setting $u_j = \max\{v_j - p_{o_j,d_j,\tau_j},\ 0\}$ for all riders $j \in \mathcal{R}$, we show that $(\pi, p, u)$ forms an optimal solution to the dual LP (16) by observing that $(\Phi, u)$ forms an optimal solution to the dual of the corresponding MCF problem (the proof of Lemma 4), and a correspondence between the optimal solutions of the dual LP (16) and the optimal solutions of the dual of the MCF (Lemma 6 in Appendix B.3). This implies that the plan determined by the STP mechanism starting from any history is individually rational for riders, strictly budget balanced, envy-free and forms a CE. The mechanism is, as a consequence, individually rational and strictly budget balanced, since the CE prices are collected from the riders and paid to the drivers only if the drivers follow the dispatches and pick up the riders.

For incentive alignment, the one deviation property for finite horizon extensive-form games [28] means that we only need show that a single deviation from the mechanism's dispatching is not

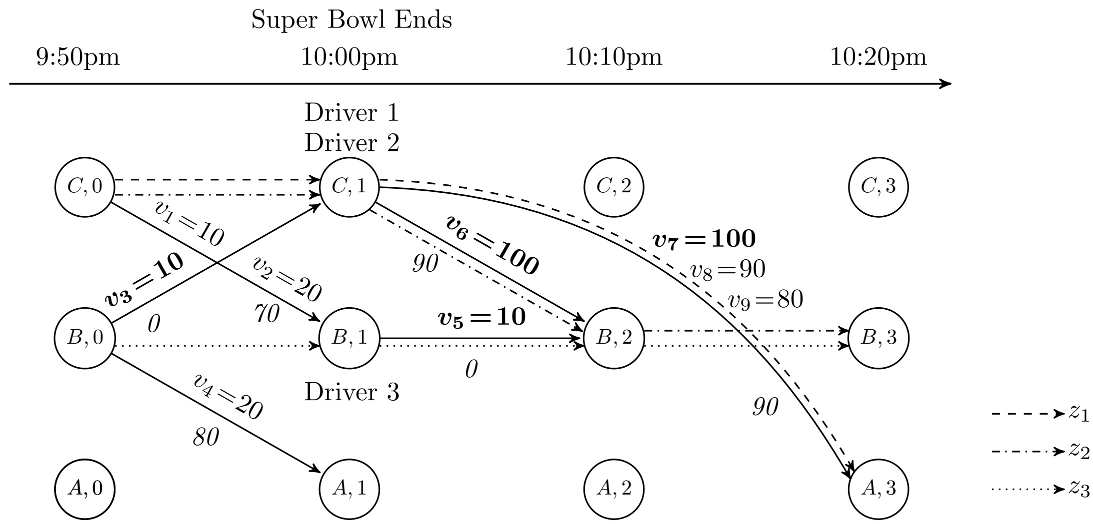



Figure 5: The Super Bowl example: replanning under the STP mechanism at time 1 after driver 3 deviated from the oroginal plan and stayed in location $B$ until time 1.

useful. The total payment to each driver from some time $t$ onward when following the current plan is equal to the welfare gain (at the time when the plan is computed) from adding a driver at her current location and time $t$. We establish that this welfare gain is weakly higher than the welfare gain for the economy (at time $t+1$) from replicating this driver at any location and time that the driver can deviate and relocate to. For this, we use the $M^\natural$ concavity (and more specifically, the local exchange properties) of optimal objectives of MCF problems [25]. In particular, to maximize welfare, there is stronger substitution among drivers at the same location than among drivers at different locations. From incentive alignment and the CE property, we have envy-freeness for drivers and riders in SPE.

**Example 2** (Continued)**.** Whereas the static CE mechanism fails to be welfare-optimal or envy-free for riders after driver 3 deviates from the plan and stays in location $B$ until time 1, the plan recomputed under the STP mechanism at time 1 is as illustrated in Figure 5. Driver 3 is be re-dispatched to pick up rider 5, and instead of picking up rider 8 whose value is 90, driver 2 now picks up rider 6 who was initially assigned to driver 3. Prices also update to $p_{C,B,1} = p_{C,A,1} = 90$, so that each driver at $(C, 1)$ is paid the welfare gain from an additional driver at $(C, 1)$. The outcome also remains envy-free for riders. The price for the $(B, B, 1)$ trip now drops to 0, since an additional driver at location $B$ at time 1 no longer contribute any welfare improvement. □

Under the STP mechanism, replanning can be triggered by the deviation of any driver, and for this reason the total payment of a driver is affected by the actions of others, and the straightforward behavior is not a dominant strategy. We show that no mechanism can implement the desired properties in a dominant-strategy equilibrium.

**Theorem 3.** *Following the mechanism's dispatch at all times does not form a dominant strategy equilibrium under any dynamic ridesharing mechanism that is, from any history onward, (i) welfare-optimal, (ii) individually rational, (iii) budget balanced, and (iv) envy-free for riders and drivers.*

*Proof.* We show that for the economy in Example 3, as shown in Figure 4, under any mechanism that satisfies conditions (i)-(iv), following the dispatches at all times cannot be a DSE. We start by analyzing what must be the outcome at time 0 under such a mechanism. At time 0, optimal welfare is achieved by dispatching one of the two drivers to go to $(B, 1)$ so that at time 1 she can pick up rider 1, and the other driver to go to $(A, 1)$ to pick up rider 2. Assume w.l.o.g. that at time 0, driver 1 is dispatched to stay in $B$ and driver 2 is dispatched to stay in $A$.

Now consider the scenario where driver 2 deviated, and took the trip $(A, B, 0)$ at time 0 instead. If driver 1 followed the mechanism's dispatch at time 0, both drivers are at $(B, 1)$ at time 1, and the welfare-optimal outcome is to pick up rider 1. Individual rationality requires that the highest amount of payment we can collect from rider 1 is 8. Budget balance and envy-freeness of drivers then imply that drivers 1 and 2 are each paid at most 4 at time 1. If driver 2 is going to deviate at time 0 and relocate to $B$, driver 1 may deviate from the mechanism's dispatch and relocate to $(A, 1)$ instead. In this case, at time 1 it is welfare optimal for driver 1 to pick up rider 2. Her payment for the trip $(A, A, 1)$ is at least 5, for otherwise rider 3 envies the outcome of rider 2. This is better than following the mechanism and get paid at most 4. □

A natural variation on the STP mechanism is to consider the driver-optimal analogue, which always computes a driver-optimal competitive equilibrium plan at the beginning of the planning horizon, or upon deviation of any driver. This mechanism pays each driver the externality she brings to the economy, $\Psi_{\ell_i, \underline{\tau}_i}$, and corresponds to the reasoning of the VCG mechanism. The driver-optimal mechanism is, however, not incentive compatible. See Appendix C.1 for a detailed example and explanation.

The STP mechanism also has the following two properties. First, the mechanism is space-time invariant, in that the driver and rider payments does not change after adding additional locations that are irreverent to rider trips, or after adding additional time periods before the beginning of the planning horizon where there is neither supply nor demand. Second, instead of assuming complete information, if the mechanism asks the drivers to report the times and locations where they will enter the platform, then for a driver with entering location and time $(\ell_i, \underline{\tau}_i)$, there is no incentive for her to report a later entrance time and location $(\hat{\ell}_i, \hat{\underline{\tau}}_i)$ s.t. $\hat{\underline{\tau}}_i \geq \underline{\tau}_i + \delta(\ell_i, \hat{\ell}_i)$, and then enter the platform at $(\hat{\ell}_i, \hat{\underline{\tau}}_i)$ (see Appendix B.6). Adopting these two additional properties, we prove the following characterization result on the uniqueness of the STP payments.

**Theorem 4** (Uniqueness of STP Payments)**.** *For any ridesharing mechanism that is SPIC, temporally consistent, and space-time invariant, if (i) the downstream plan from any history onward is welfare-optimal and forms a CE, and (ii) drivers do not have incentives to delay entrance to the platform, then the driver payments must be the same as those under the STP mechanism.*

Intuitively, if there exists an economy where some driver is paid higher than the driver-pessimal payment, we can construct a larger economy, where there exists a driver who either has incentives to deviate from the mechanism's dispatches, or to delay entrance to the platform. Payments lower than the driver-pessimal do not form CE, therefore the STP payments are unique.[20] See Appendix B.6 for the proof of this theorem.

## 4.3 Discussion

Throughout the paper, we assume (S1) that drivers stay until the end of the planning horizon and have no intrinsic preference over locations, and (S2), that riders are impatient. Examples 6 and 7 in Appendix C.2 show that if either of these assumptions is relaxed, the LP relaxation (10) of the planning problem may no longer be integral, and thus, there may not exist anonymous,

[20]Instead of CE, if we assume that the mechanism uses anonymous trip prices and satisfies from any history onward IR and envy-freeness on the drivers' side, then we can show that no driver can be paid any payment higher that that under the STP mechanism. See Appendix B.6.

origin-destination CE prices. This is because the reduction to the minimum cost flow problem fails (see Appendix C.2).

The triangle inequality, $\delta(a,b) \leq \delta(a,c) + \delta(b,c)$ for all $a,b,c \in \mathcal{L}$, is not necessary for our results on drivers' incentives. On the rider's side, one concern in practice is that riders may try to break a long trip into a few shorter trips in order to get a lower total price, especially for platforms where riders may set multiple stops a trip.[21] With the triangle inequality, we recover an arbitrage-proofness: the total price under the STP mechanism for the shorter trips is higher than for the single trip. This is because the shorter trips take a longer total time, and the rider's total payment, which equals the difference in the welfare gain from having an additional driver at the (origin, starting time) and the (destination, ending time) of the trip, is higher if the ending time is later.

The focus of this paper is to align incentives for drivers in a dynamic environment while maintaining driver flexibility, instead of that of information asymmetry. On the rider side, we show via Example 8 in Appendix C.3 that although the STP mechanism chooses a driver-pessimal CE plan, it may not be a dominant strategy for the riders to truthfully report their values. This is different from the classical, unit-demand assignment problem [33], where the set of CE prices form a lattice— the seller-pessimal CE prices correspond to the buyer-side VCG prices, and is incentive compatible for the buyers. In fact, we show in Theorem 6 that the rider-side VCG payment for a rider is equal to the minimum price for her trip among all CE outcomes, and such trip prices may not form a CE. This implies that no mechanism that computes optimal CE plans and balances budget incentivizes the riders to truthfully report their values. Moreover, we show that no BB, optimal and SPIC mechanism is incentive compatible on the rider's side.

On the driver's side, we may consider a model where the mechanism asks the drivers to report their times and locations to enter the platform, before planning at the beginning of the planning horizon. We show in Appendix B.6 that under the STP mechanism, for a driver with $(\ell_i, \underline{\tau}_i)$, there is no incentive for her to report a later entrance time and location $(\hat{\ell}_i, \hat{\underline{\tau}}_i)$ s.t. $\hat{\underline{\tau}}_i \geq \underline{\tau}_i + \delta(\ell_i, \hat{\ell}_i)$, and then enter the platform at $(\hat{\ell}_i, \hat{\underline{\tau}}_i)$. If drivers are allowed to enter at any arbitrary time and location regardless of their reports, and if the mechanism simply replans without penalizing the drivers that deviated from their reports, we also show that no mechanism with the economic properties of the STP mechanism incentivizes the drivers to truthfully report their entrance information. We consider it less important to provide flexibility for drivers to enter the platform in a way that is different from their reports, since there is less uncertainty in driver's ability to enter (e.g. many drivers start driving from home at a certain time).

# 5 Simulation Results

In this section, we compare, through numerical simulation, the performance of the STP mechanism against the myopic pricing mechanism, for three stylized scenarios: the end of a sporting event, the morning rush hour, and trips to and from the airport with imbalanced demand.

In addition to social welfare, we consider the *time-efficiency* of drivers, which is defined as the proportion of time where the drivers have a rider in the car, divided by the total time drivers spend on the platform. We also consider the *regret* to drivers for following the straightforward strategy in a non incentive-aligned mechanism: the highest additional amount a driver can gain by strategizing in comparison to following a mechanism's dispatch, assuming that the rest of the drivers all follow the mechanism's dispatches at all times. The analysis suggests that the STP mechanism achieves substantially higher social welfare, as well as time-efficiency for drivers, whereas, under the myopic pricing mechanism, drivers incur a high regret.

[21] `https://www.ridester.com/uber-adds-multiple-stops/`, visited January 8, 2018.

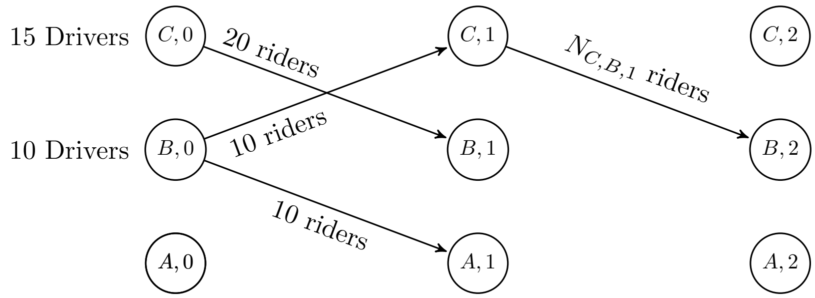
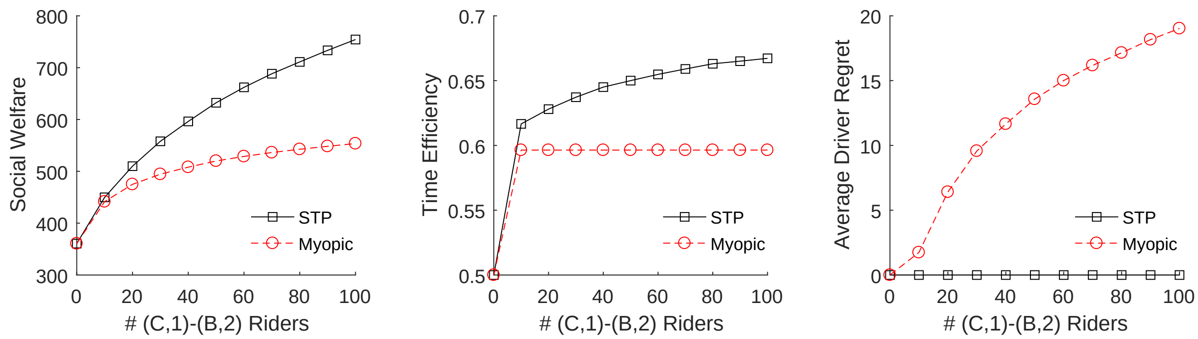



Figure 6: An example to illustrate the end of an event.




(a) Social welfare.

(b) Driver time efficiency.

(c) Average driver regret.

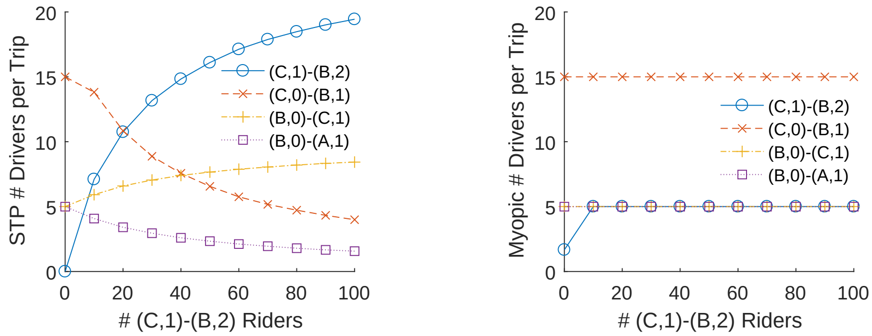
Figure 7: Comparison of welfare, driver time efficiency and average driver regret for the of an event.

We define the myopic pricing mechanism to use the lowest market clearing prices (which market clearing prices are chosen is unimportant for the results). In addition, because this mechanism need not always dispatch all available drivers, we model any available driver who is not dispatched as randomly relocating to a location that is within reach by the end of the planning horizon.

### 5.1 Scenario One: The End of a Sporting Event

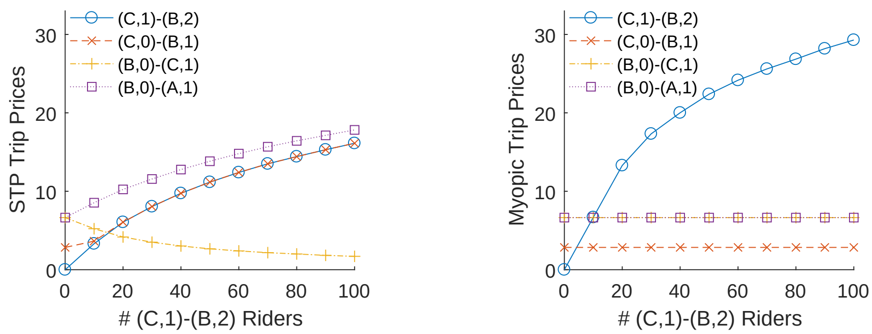
We first consider the scenario in Figure 6, modeling the end of a sporting event. There are three locations $\mathcal{L} = \{A, B, C\}$ with unit distances $\delta(a, b) = 1$ for all $a, b \in \mathcal{L}$, and two time periods. The event ends at location $C$ at time 1, where there would be a large number of riders requesting rides. In each economy, at time 0, there are 15 and 10 drivers starting at locations $C$ and $B$. 20 riders request trip $(C, B, 0)$ and 10 riders request trips $(B, C, 0)$ and $(B, A, 0)$ respectively. When the event ends, there are $N_{C,B,1}$ riders hoping to take a ride from $(C, 1)$ to $(B, 2)$. The values of all riders are independently drawn from the exponential distribution with mean 10.

As we vary the number of riders $N_{C,B,1}$ requesting the trip $(C, B, 1)$ from 0 to 100, we randomly generate $1,000$ economies, and compare the average welfare, driver's time efficiency and average regret as shown in Figure 7. Figure 7a shows that the STP mechanism achieves a substantially higher social welfare than the myopic pricing mechanism, especially when there are a large number of drivers taking the trip $(C, B, 1)$. Figure 7b shows that the STP mechanism becomes more time-efficient as the number of $(C, B, 1)$ riders increases, as more of the 10 drivers starting at $B$ takes the $(B, C, 0)$ trip and then pick up another rider at $(C, 1)$. We also see the extent to which the myopic pricing mechanism is not incentive aligned. Drivers that are dispatched to the trips $(C, B, 1)$ and




(a) The STP mechanism. (b) The myopic pricing mechanism.

Figure 8: Comparison of the number of drivers per trip for the end of an event.




(a) The STP mechanism. (b) The myopic pricing mechanism.

Figure 9: Comparison of trip prices for the end of an event.

$(B, A, 1)$ may regret having not relocated to $C$ instead and get paid a large amount at time 1. Figure 7c shows that the average regret of the 25 drivers increases significantly as $N_{C,B,1}$ increases.

The average number of drivers taking each of the four trips of interest under the two mechanisms are shown in Figure 8. As $N_{C,B,1}$ increases, the STP mechanism dispatches more drivers to $(C, 1)$ to pick-up higher-valued riders leaving $C$, while sending less drivers on trips $(C, B, 0)$ and $(B, A, 0)$. The myopic pricing mechanism, being oblivious to future demand, sends all drivers starting at $(C, 0)$ to location $B$, and an average of only 5 drivers to $(C, 1)$ from $(B, 1)$.

The average trip prices are plotted in Figure 9. First of all, prices are temporally "smooth"— trips leaving $C$ at times 0 and 1 have very similar prices. Moreover, the prices for trips $(B, C, 0)$ and $(C, B, 1)$ add up to the price of the trip $(B, A, 0)$, so that drivers starting at $(B, 1)$ would not envy each other. In contrast, prices for trips leaving $C$ increase significantly under the myopic pricing mechanism, and the drivers taking the trip $(B, A, 0)$ envy those that take $(B, C, 0)$ and subsequently $(C, B, 1)$. The "surge" for the trip $(C, B, 1)$ is significantly higher under the myopic pricing mechanism, implying that the platform is providing even less price reliability for the riders.

## 5.2 Scenario Two: The Morning Rush Hour

We now compare the two mechanisms for the economy in Figure 10, modeling the demand pattern of the morning rush hour. There are $T = 10$ time periods and three locations $\mathcal{L} = \{A, B, C\}$ with $\delta(a, b) = 1$ for all $a, b \in \mathcal{L}$. $C$ is a residential area, where there are a number of riders requesting

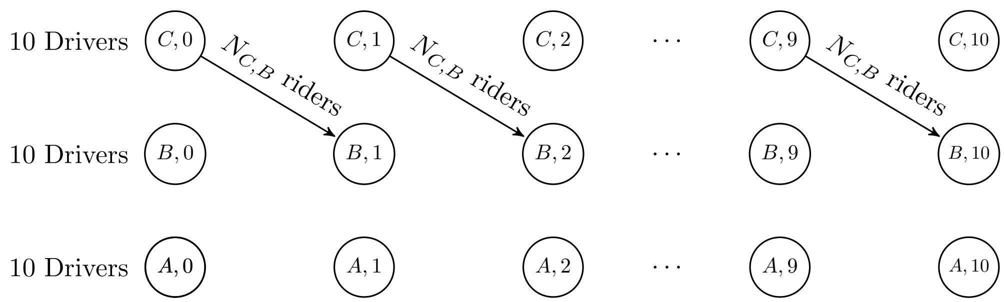
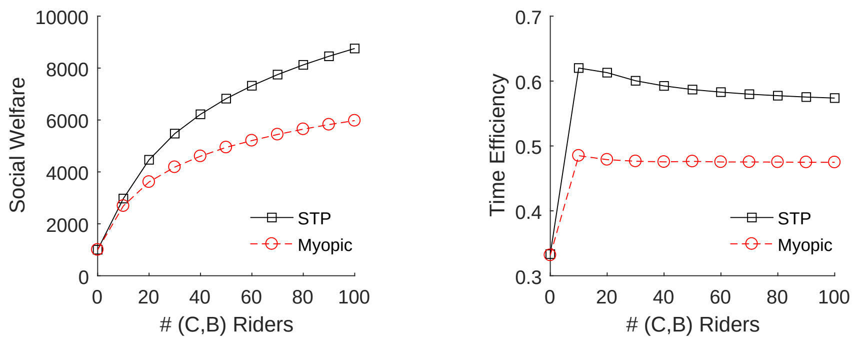



Figure 10: Morning rush hour.




(a) Social welfare.

(b) Driver time efficiency.

Figure 11: Comparison of social welfare and driver time efficiency for the morning rush hours.

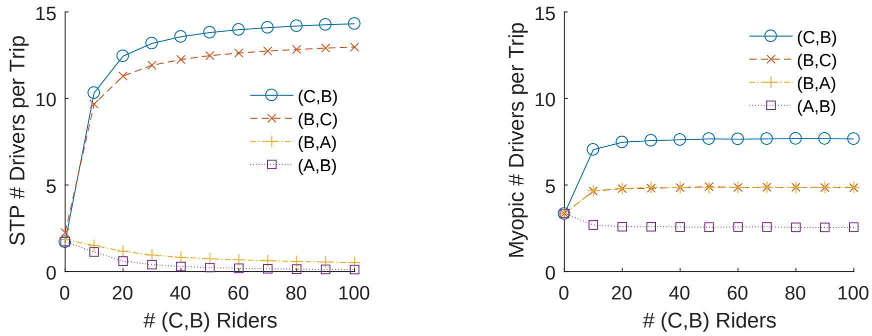
rides to $B$, the downtown area, in every period. Location $A$ models some other area in the city.

In each economy, for each location $A$, $B$ and $C$, there are 10 drivers entering at time 0, who all stay until the end of the planning horizon. There are a total of 100 riders with trip origins and destinations independently drawn at random from $\mathcal{L}$ and trip starting times randomly drawn from $[T-1]$. In addition, in each period there are $N_{C,B}$ commuters traveling from $C$ to $B$. We assume that the commuters' values for the rides are i.i.d. exponentially distributed with mean 20, whereas the random rides have values exponentially distributed with mean 10.

As we vary the $N_{C,B}$ from 0 to 100, the average social welfare achieved by the two mechanisms for $1,000$ randomly generated economies is as shown in Figure 11a. The STP mechanism achieves higher social welfare than the myopic pricing mechanism. Figure 11b shows that the STP mechanism achieves much higher driver time efficiency. The time efficiency of both mechanisms actually decreases as the number of $(C, B)$ riders per period increases above 10. For the STP mechanism, this is because the mechanism sends more empty cars to $C$ to pick up the higher value riders there. For the myopic pricing mechanism, it is because the large number of $(C, B)$ trips brings too much excessive supply at $B$, which are not utilized efficiently in the subsequent periods.

For the four origin-destination pairs, $(C, B)$, $(B, C)$, $(B, A)$ and $(A, B)$, Figures 12 and 13 plot the average number of drivers getting dispatched to take these trips in each time period (including both trips with a rider, and repositioning without a rider), and the average trip prices. For each economy, the number of drivers for each origin-destination (OD) pair and the trip prices for this OD pair are averaged over the entire planning horizon. Results on the other five trips, $(A, A)$, $(A, C)$, $(B, B)$, $(C, A)$ and $(C, C)$ can be interpreted similarly, therefore are deferred to Appendix E.

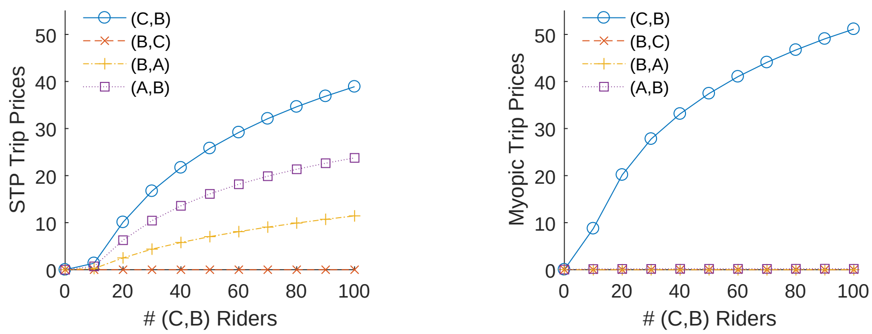
Under the STP mechanism, given the large demand for trips from $C$ to $B$ in each time period, there is a large number of drivers taking the trip $(C, B)$, and also a large number of drivers relocating




(a) The STP mechanism.

(b) The myopic pricing mechanism.

Figure 12: Comparison of the number of drivers per trip for the morning rush hour.




(a) The STP mechanism.

(b) The myopic pricing mechanism.

Figure 13: Comparison of trip prices for the morning rush hour.

from $B$ to $C$ in order to pick up future riders from $C$ (see Figure 12a). A small number of drivers are dispatched from $B$ to $A$ due to the lack of future demand at $A$. Because of the abundance of supply at $B$ that are brought in by the $(C, B)$ trips, very few drivers are sent from $A$ to $B$.

Under the myopic pricing mechanism, the number of drivers dispatched to take each trip, in contrast, does not contribute to the repositioning of drivers. See Figure 12b. There are an equal number of drivers traveling from $B$ to $A$ and $C$ despite the significant difference in the demand conditions at the two destinations. Moreover, a non-trivial number of drivers are traveling from $A$ to $B$ despite the fact that there are already too much of supply at location $B$.

Regarding the average prices under the STP mechanism plotted in Figure 13a, the morning commute route $(C, B)$ has a higher average price due to the large demand for this trip. The $(B, A)$ trip is less costly than the $(A, B)$ trip since there is plenty of supply of drivers that are brought to $B$ by the $(C, B)$ trips, so that the marginal value of supply at $B$ is low. The $(A, B)$ price is high so that not too many drivers are dispatched from $A$ to $B$. The $(B, C)$ trips are priced almost always at zero since it is beneficial for the economy for drivers to be at $C$ to pick up the commuters.

Finally, Figure 13b shows that the $(C, B)$ trip has a much higher average price under the myopic pricing mechanism than the STP mechanism, whereas the rest three trips are priced at zero— this is because without optimizing for the supply of drivers at each location, locations $A$ and $B$ almost always have plenty of supply to pick up all riders starting from these locations, whereas there is far from enough drivers to satisfy the large demand at location $C$.

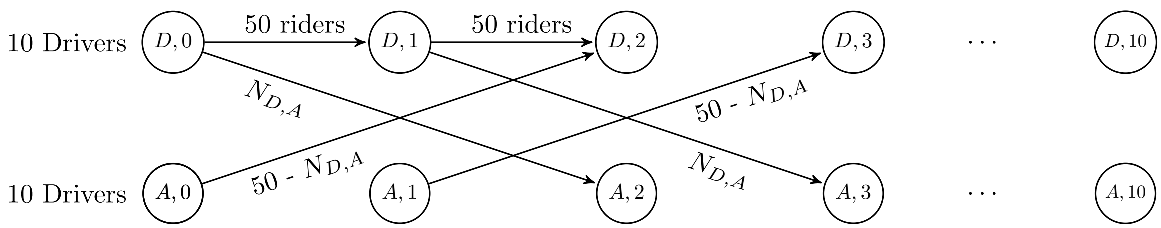
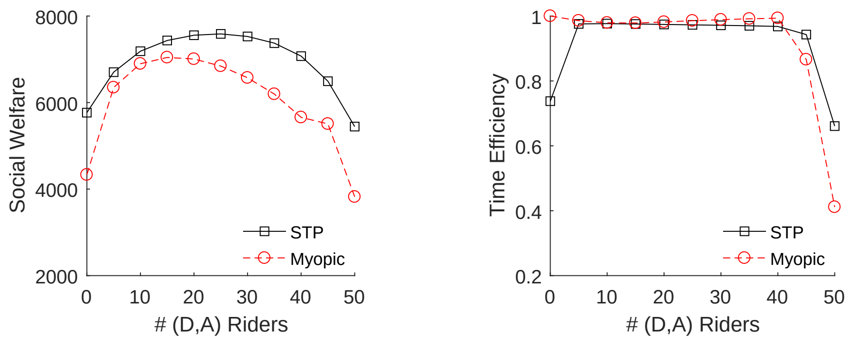



Figure 14: Imbalance in trips to and from the airport.




(a) Social welfare.

(b) Driver time efficiency.

Figure 15: Comparison of social welfare and driver time efficiency for the morning rush hours.

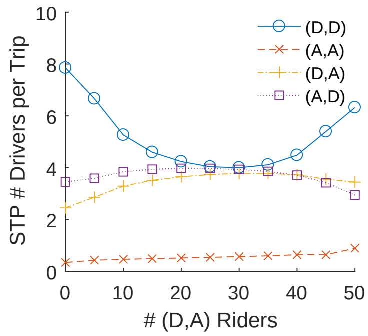
### 5.3 Scenario Three: Imbalanced Airport Trips

In this scenario, we consider the imbalance between trips to and from the airport from the downtown area, as illustrated in Figure 14. There are a total of $T = 10$ time periods and two locations $\mathcal{L} = \{A, D\}$, modeling the airport and the downtown area respectively. $\delta(A, A) = \delta(D, D) = 1$, whereas trips in between downtown and the airport are longer: $\delta(A, D) = \delta(D, A) = 2$.

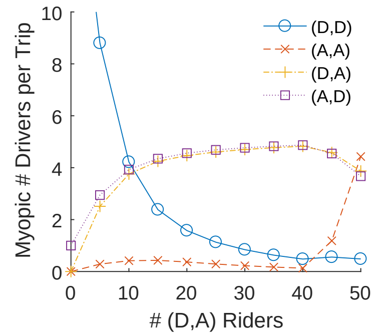
In each economy, there are 10 drivers entering at each of $A$ and $D$ at time 0. Within the downtown area, there are 50 riders requesting rides in each period, whereas in between the downtown area and the airport, there are a total of 50 riders heading toward or leaving the airport, which may be unevenly distributed on the two directions. The value of each of the downtown riders are drawn i.i.d from the exponential distribution with mean 10, and the value for each trip to or from the airport is drawn i.i.d from the exponential distribution with mean 30. Since the airport trips are twice as long, we are modeling the scenario where the airport travelers are less price sensitive, and are willing to pay on average 50% higher than the downtown riders.

As we vary the number of $(D, A)$ riders $N_{D,A}$ from 0 to 50 (thus at the same time varying the number of $(A, D)$ riders from 50 to 0), the average social welfare and driver-time efficiency achieved by the two mechanisms over 1,000 randomly generated economies are as shown in Figure 15. We first observe that the more balanced the trip flow to and from the airport is (i.e. when $N_{D,A}$ is closer to 25), the higher the social welfare achieved by STP is, although the driver-time efficiency is mostly close to 1. This is because when the trip flow is more balanced, it is more likely for a driver to pick up riders with high values for the trips both to and from the airport, whereas when the flow is imbalanced, drivers may relocate with an empty car or pick up riders with low values for one of the two directions.

The welfare achieved by the myopic pricing mechanism, however, is highly asymmetric. This is because riders starting from the downtown area $D$ are ordered by their values, and the airport trips

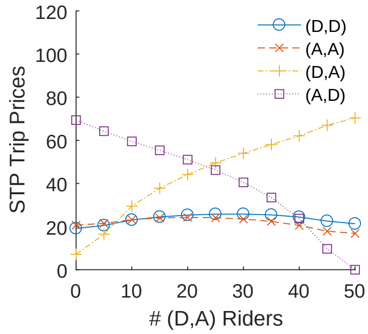



(a) The STP mechanism.

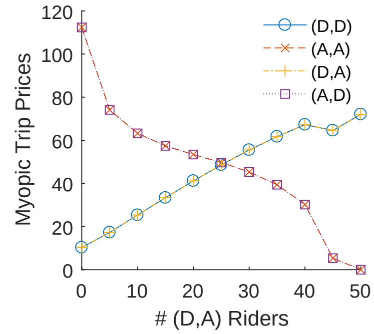



(b) The myopic pricing mechanism.

Figure 16: Comparison of driver numbers for the imbalanced trips to and from the airport.




(a) The STP mechanism.




(b) The myopic pricing mechanism.

Figure 17: Comparison of trip prices for the imbalanced trips to and from the airport.

are longer and have higher values. Figure 16b shows that as the number of $(D, A)$ riders increases, the myopic pricing mechanism quickly ends up sending a good number of drivers to the airport, and achieving good welfare when the number of $(D, A)$ riders is only around 15.

As $N_{D,A}$ keeps increasing, too many drivers are dispatched to the $(D, A)$ trip, which in turn results in too many drivers moving from $A$ back to $D$ driving riders with low values. Too few of the riders traveling within the downtown area are actually picked up, resulting in the low social welfare when the $N_{D,A}$ is high. Comparing Figures 16a and 16b, we also see that the STP mechanism completes more $(D, D)$ trips when the airport trip flows are more imbalanced, and less $(D, D)$ trips when the flow is more balanced, therefore making the airport trips more efficient.

The average prices for trips under the two mechanisms are shown in Figure 17. The STP mechanism sets prices for the $(D, D)$ trip that is almost symmetric, higher when the airport trip flow is more balanced. The price for the overly-demanded direction is increased whereas the price for the direction with little demand is low, in order to moderate a more efficient driver trip flow (i.e. sending fewer drivers to the airport if there are too few riders leaving the airport). The airport trip prices under the myopic pricing mechanism also exhibit similar patterns. However, the $(D, D)$ trip is priced identically to the $(D, A)$ trip, forgoing lots of reasonably high valued downtown riders whose trips take only 1 period of time.

# 6 Conclusion

We study the problem of optimal dispatching and pricing in two-sided ridesharing platforms in a way that drivers would choose to accept the platform's dispatches instead of driving to another area or waiting for a higher price. Under a complete information, discrete time, multi-period and multi-location model, we show that always following the mechanism's dispatches forms a subgame-perfect equilibrium among the drivers under the spatio-temporal pricing mechanism, which always computes a driver-pessimal competitive equilibrium plan at the beginning of the planning horizon as well as after any deviations. Our empirical study suggests that the STP mechanism achieves substantially higher social welfare and drivers' time efficiency in comparison to the myopic pricing mechanism, where in addition drivers incur a high regret.

**Future Work** Throughout the paper, we assumed complete information and a finite planning horizon. Our ongoing work include generalizing the model to settings where there is uncertainty in demand/supply prediction, and where the planning horizon rolls forward as the uncertainty gets resolved over time. One challenge is that of balancing budget— strict balance does not hold for arbitrary driver action profiles, but may still hold in subgame perfect equilibrium. Another challenge is that the class of $M^\natural$ concave functions is not closed under addition, thus the continuation value of a distribution of drivers at the end of the planning horizon (which is the expectation of its continuation value for each realized future demand pattern) may no longer be $M^\natural$ concave, and this affects the integrality of the optimal planning problem and the existence of CE.

Other interesting directions of future work include: (i) empirical analysis on the ridesharing platforms, especially on the loss of welfare due to myopia and the strategic behavior of drivers, (ii) the fairness in regard to effect of welfare-optimal planning on trip prices to and from different neighborhoods with different demand patterns, (iii) truthful elicitation of information in scenarios where drivers poses information on local demand/supply in the near future, and where drivers have preferences with respect to locations or heterogeneity in costs, and (iv) pricing and dispatching with multiple classes of services, and with the existence of competition between platforms.

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# Appendix

Appendix A provides a continuous-time interpretation of the discrete time model that we adopted. Appendix B includes proofs that are omitted from the body of the paper. Appendix C provides examples and discussions on the driver-optimal mechanism, integrality of the LP relaxation and existence of CE, incentives of riders, and the naive recomputation of optimal CE plans. Appendix D discusses the relationship with the literature on trading networks and the dynamic VCG mechanism, and why they do not solve the ridesharing problem. Finally, additional simulation results are presented in Appendix E.

## A Continuous Time Interpretation

Under the discrete time model introduced in Section 2, trips within the same location takes $\delta(a,a) = 1$ unit of time for all $a \in \mathcal{L}$, however, we also assume that a driver can drop-off a rider and pick-up a new rider in the same location at the same time point.

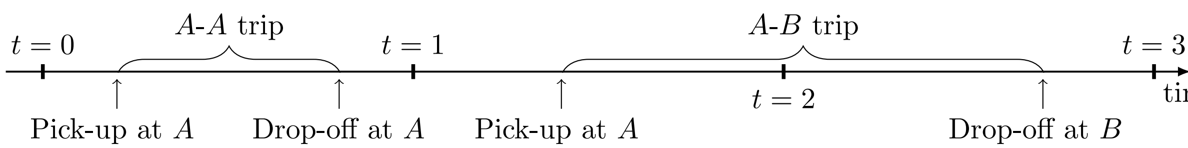



Figure 18: Time-line for a within-location trip $A \to A$ which takes $\delta(A,A) = 1$ period of time, and a between-location trip $A \to B$ which takes $\delta(A,B) = 2$ periods of time.

Figure 18 illustrates the continuous-time interpretation of this discrete-time model. There are two trips: an $A$ to $A$ trip at time 0 which takes $\delta(A,A) = 1$ unit of time, and an $A$ to $B$ trip at time 1, which takes $\delta(A,B) = 2$ units of time and ends at time $t = 3$. The time after the drop-off of the first rider at $A$ and the time for the second pick-up at $A$ is the time the driver takes to travel within location $A$ to pick up the second rider. In the discrete time model, both the drop-off and the pick-up are both considered to happen at time $t = 1$.

## B Proofs

### B.1 Proof of Lemma 1

**Lemma 1.** Given any CE plan $(x, \tilde{z}, p)$, the plan with anonymous prices $(x, \tilde{z}, p^+)$ also forms a CE, and has the same driver and rider payments as those under the original CE plan $(x, \tilde{z}, p)$.

*Proof.* We first show that given any CE plan $(x, \tilde{z}, p)$, for any trip that is requested by any rider, the anonymous trip prices must be non-negative. Assume toward a contradiction, that there exists $(a,b,t) \in \mathcal{T}$ s.t. $\exists j \in \mathcal{R}$ s.t. $(o_j, d_j, \tau_j) = (a,b,t)$ and $p_{a,b,t} < 0$. Rider best response implies $x_j = 1$, thus there exists $i \in \mathcal{D}$ s.t. $(o_j, d_j, \tau_j, j) \in \tilde{z}_i$, and this is paid $p_{a,b,t} < 0$ at time $t$. This violates driver best response, since keeping the rest of the action path unchanged, but choosing not to bet paid for this trip, the driver would get a higher total payment.

This implies that $p_{o_j,d_j,\tau_j} = p^+_{o_j,d_j,\tau_j}$ holds for all $j \in \mathcal{R}$, therefore given the dispatching $(x, \tilde{z})$, rider best response under $p$ implies rider best response under $p^+$. Moreover, for each driver $i \in \mathcal{D}$, $\pi_i = \sum_{j \in \mathcal{R}} \mathbb{1}\{(o_j, d_j, \tau_j, j) \in \tilde{z}_i\} p_{o_j,d_j,\tau_j} = \sum_{j \in \mathcal{R}} \mathbb{1}\{(o_j, d_j, \tau_j, j) \in \tilde{z}_i\} p^+_{o_j,d_j,\tau_j} = \max_{z_i \in \mathcal{Z}_i} \left\{ \sum_{(a,b,t) \in z_i} \max\{p_{a,b,t}, 0\} \right\} = \max_{z_i \in \mathcal{Z}_i} \left\{ \sum_{(a,b,t) \in z_i} \max\{p^+_{a,b,t}, 0\} \right\}$, therefore driver best response also holds. This completes the proof that $(x, \tilde{z}, p)$ also forms a CE, and that the driver and rider payments under the two plans are identical. $\square$

## B.2 Proof of Lemma 2

### B.2.1 Minimum Cost Flow

The technical results in this paper made extensive use of the reduction of an optimal dispatching problem for ridesharing to a minimum cost flow (MCF) problem, where drivers flow through a network, with nodes corresponding to (location, time) pairs, and the edge costs equal to the negations (additive inverse) of rider values. See [2] for an introduction to MCF. We briefly describe notations and the formulation of the MCF problems in this section.

Let $G = (\mathcal{N}, \mathcal{E})$ be a directed graph with a node set $\mathcal{N}$ and an edge set $\mathcal{E}$. Let $\underline{c} : \mathcal{E} \to \mathbb{Z} \cup \{-\infty\}$ be the lower capacity function, $\bar{c} : \mathcal{E} \to \mathbb{Z} \cup \{+\infty\}$ be the upper capacity function, and let $\gamma : \mathcal{E} \to \mathbb{R}$ be the cost function. For each edge $e \in \mathcal{E}$, denote $\partial^+ e \in \mathcal{N}$ as the initial (tail) node of $e$ and $\partial^- e \in \mathcal{N}$ as the terminal (head) node of $e$. That is, $\partial^+ e = n_1$ and $\partial^- e = n_2$ for the edge $e = (n_1, n_2)$.

A feasible flow $f$ is a function $f : \mathcal{E} \to \mathbb{R}$ such that $\underline{c}(e) \le f(e) \le \bar{c}(e)$ for each $e \in \mathcal{E}$. Its boundary $\partial f : \mathcal{N} \to \mathbb{R}$ is defined as

$$\partial f(n) = \sum \{f(e) \mid e \in \mathcal{E},\ \partial^+ e = n\} - \sum \{f(e) \mid e \in \mathcal{E},\ \partial^- e = n\}.$$

A node $n$ for which $\partial f(n) > 0$ is a source of the flow, and a node $n$ is a sink if $\partial f(n) < 0$. Let $\xi$ be a vector in $\mathbb{R}^{|\mathcal{N}|}$, the minimum cost for any flow with boundary condition $\xi$ is:

$$\omega(\xi) = \inf_f \left\{ \sum_{e \in \mathcal{E}} \gamma(e) f(e) \,\middle|\, f : \text{ feasible flow with } \partial f = \xi \right\}. \tag{31}$$

**Optimal Dispatching $\Rightarrow$ MCF** Given an instance of the optimal dispatching problem with planning horizon $T$, locations $\mathcal{L}$, distances $\delta$, riders $\mathcal{R}$ and drivers $\mathcal{D}$, we construct a corresponding MCF problem. Let $G = (\mathcal{N}, \mathcal{E})$ be the graph, where the nodes $\mathcal{N}$ consists of (location, time) pairs with an additional "sink" node $n_0$ representing the end of time:

$$\mathcal{N} = \{ (a, t) \mid a \in \mathcal{L},\ t \in [T]\} \cup \{n_0\}.$$

The set of edges $\mathcal{E} = \mathcal{E}_1 \cup \mathcal{E}_2 \cup \mathcal{E}_3$ consists of the following three parts:

- $\mathcal{E}_1 = \{ e_j \mid j \in \mathcal{R}\}$ corresponds to rider trips, where the edge $e_j = ((o_j, \tau_j),\ (d_j, \tau_j + \delta(o_j, d_j)))$ corresponds to the trip requested by rider $j$ and has minimum capacity $\underline{c}(e_j) = 0$, maximum capacity $\bar{c}(e_j) = 1$ and cost $\gamma(e_j) = -v_j$. Intuitively, if a unit of driver flows through the edge corresponding to rider $j$ (i.e. rider $j$ is picked up by a driver), we incur a cost of $-v_j$, which is gaining value $v_j$.

- $\mathcal{E}_2$ consists of edges that are feasible relocating trips without riders:

  $$\mathcal{E}_2 = \{ ((a, t),\ (b, t + \delta(a, b)) \mid (a, b, t) \in \mathcal{T}\}.$$

  Recall that $\mathcal{T} = \{(a, b, t) | a, b \in \mathcal{L}, t + \delta(a, b) \le T\}$ denotes the set of feasible trips within the planning horizon. There is no upper bound on capacities of these edges, and the edge costs are zero: $\forall e \in \mathcal{E}_2$, $\underline{c}(e) = 0$, $\bar{c}(e) = +\infty$, and $\gamma(e) = 0$.

- $\mathcal{E}_3$ consists of edges that connect nodes at time $T$ to the sink:

  $$\mathcal{E}_3 = \{ ((a, T),\ n_0) \mid a \in \mathcal{L}\}.$$

  Similar to $\mathcal{E}_2$, $\forall e \in \mathcal{E}_3$, $\underline{c}(e) = 0$, $\bar{c}(e) = +\infty$, and $\gamma(e) = 0$.

**Flow LP** The boundary condition of the MCF problem constructed from a ridesharing instance is given by the time and location where drivers enter the platform. Fix a node $n = (a, t)$ for some $a \in \mathcal{L}$ and $t \in [T]$, $\xi_n = \sum_{i \in \mathcal{D}} \mathbb{1}\{\ell_i = a,\ \underline{\tau}_i = t\}$, i.e. the units of driver flow originating from node $n$ is equal to the number of drivers entering at $n$. For the sink node $n_0$, $\xi_{n_0} = -m$, where $m$ is the number of drivers.

Given this construction, there are non-zero edge costs and upper flow capacity constraints only for edges in $\mathcal{E}_1$. The minimum cost flow problem (31) can therefore be simplified and rewritten in the following form:

$$\min_f \sum_{j \in \mathcal{R}} -v_j f(e_j) \tag{32}$$

$$\text{s.t.} \sum_{e \in \mathcal{E},\ \partial^+ e = (a,t)} f(e) - \sum_{e \in \mathcal{E},\ \partial^- e = (a,t)} f(e) = \sum_{i \in \mathcal{D}} \mathbb{1}\{\ell_i = a,\ \underline{\tau}_i = t\}, \quad \forall a \in \mathcal{L},\ \forall t \in [T] \tag{33}$$

$$f(e_j) \leq 1,\ \forall j \in \mathcal{R} \tag{34}$$

$$f(e) \geq 0,\ \forall e \in \mathcal{E} \tag{35}$$

Note that given (33), the flow balance constraint at the sink, $\sum_{a \in \mathcal{L}} f(((a, T), n_0)) = m$, is redundant, and therefore omitted from the above formulation in order to achieve better interpretability of the dual variables. Observing that minimizing the negation of the total value of riders that are picked up is equivalent to maximizing the total value of riders that are picked up, we can rewrite (32) in the following form:

$$\max_f \sum_{j \in \mathcal{R}} v_j f(e_j) \tag{36}$$

$$\text{s.t.} \sum_{e \in \mathcal{E},\ \partial^+ e = (a,t)} f(e) - \sum_{e \in \mathcal{E},\ \partial^- e = (a,t)} f(e) = \sum_{i \in \mathcal{D}} \mathbb{1}\{\ell_i = a,\ \underline{\tau}_i = t\}, \quad \forall a \in \mathcal{L},\ \forall t \in [T] \tag{37}$$

$$f(e_j) \leq 1,\ \forall j \in \mathcal{R} \tag{38}$$

$$f(e) \geq 0,\ \forall e \in \mathcal{E} \tag{39}$$

We refer to (36) as the flow LP.

#### B.2.2 Proof of Lemma 2

**Lemma 2** (Integrality)**.** There exists an integer optimal solution to the linear program (10).

*Proof.* It is known that the MCF problems with certain structure have integral optimal solutions [25]. The flow LP (36) is integral since (I) the flow balance constraint (37) can be written in matrix form $Ff = \xi$ where $F$ is total unimodular and $\xi$ has only integer entries and (II) the edge capacity constraints (38) are all integral. See Section III.1.2 in [35] for details on total unimodularity and the integrality of polyhedron.

To prove the integrality of the original LP (10), we show that

(i) for each feasible solution to the LP (10), there exists a feasible solution to the flow LP (36) with the same objective, and

(ii) for each integral feasible solution to the flow LP (36), there a corresponding integral feasible solution to the LP (10) with the same objective.

The integrality of MCF then implies that there exists an integral optimal solution of (10), since the optimal objective of the LP (10) cannot exceed the optimal objective of the MCF, which is achieved at some integral feasible solution of the MCF, and therefore also at some integral feasible solution of (10). We now prove (i) and (ii).

*Part (i).* Let $(x, y)$ be a feasible solution to the LP (10). For each $i \in \mathcal{D}$ s.t. $\sum_{i,k} y_{i,k} < 1$, it is without loss to increase $y_{i,k}$ for some arbitrary $k$ s.t. the constraint $\sum_{i,k} y_{i,k} \leq 1$ is binding, thus we assume that (12) is binding for all $i \in \mathcal{D}$. A solution $f$ to the flow LP can be constructed as follows:

- For each $j \in \mathcal{R}$, let $f(e_j) = x_j$. We know that $0 \leq f(e_j) \leq 1$ for all $j \in \mathcal{R}$.
- For each $e = ((a, t),\ (b, t + \delta(a, b)) \in \mathcal{E}_2$ corresponding to the relocation trip $(a, b, t) \in \mathcal{T}$, let $f(e) = \sum_{i \in \mathcal{D}} \sum_{k=1}^{|\mathcal{Z}_i|} y_{i,k} \mathbb{1}\{(a, b, t) \in Z_{i,k}\} - \sum_{j \in \mathcal{R}} x_j \mathbb{1}\{(o_j, d_j, \tau_j) = (a, b, t)\}$, i.e. the total number of drivers whose paths cover $e$, minus the total number of riders that are picked up for this trip. The feasibility constraint (11) guarantees that $f(e) \geq 0$.
- For each $e = ((a, T), n_0) \in \mathcal{E}_3$, let $f(e) = \sum_{e' \in \mathcal{E}_1 \cup \mathcal{E}_2,\ \partial^- e' = (a,T)} f(e')$ to balance the flow in and out of $(a, T)$.

The edge capacity constraints (38) and (39) are satisfied by construction. Given the assumption that (12) is binding for all $i \in \mathcal{D}$ and the fact that each $Z_{i,k}$ is a feasible path originating from the the node $(\ell_i, \underline{\tau}_i)$, the flow balance constraints (37) are satisfied. Moreover, the objectives of the two linear programs coincide: $\sum_{j \in \mathcal{R}} x_j v_j = \sum_{j \in \mathcal{R}} v_j f(e_j)$. Therefore, $f$ is a feasible solution to the flow LP with the same objective.

*Part (ii).* Given a feasible, integral solution $f$ to the flow LP (36), we construct an integral feasible solution to the original LP. For the riders, let $x_j = f(e_j)$ for all $j \in \mathcal{R}$. For the drivers, from the standard flow decomposition arguments [8], the $m$ units of flow in $f$ that all converge in $n_0$ can be decomposed into $m$ paths of single units of flow, that correspond to each driver's feasible path taken over the entire planning horizon. This gives us a feasible solution to the original LP with the same objective, and this completes the proof of the lemma. □

The reduction to MCF can also be used to solve the original LP efficiently. In the optimal dispatching problem, the number of feasible paths for each driver is exponential in $|\mathcal{L}|$ and $T$, thus there are exponentially many decision variables in the LP (10). The numbers of decision variables and constraints of the flow LP are, in contrast, polynomial in $|\mathcal{R}|$, $|\mathcal{L}|$ and $T$, and there are efficient algorithms for solving network flow problems (see [2]).

### B.3 Proof of Lemma 3

Before proving the lemma, we first state the complementary slackness (CS) conditions [8]. Given a feasible solution $(x, y)$ to the primal LP (10), and a feasible solution $(p, \pi, u)$ to the dual LP (16), both solutions are optimal if and only if the following conditions hold:

(CS-1) for all $j \in \mathcal{R}$, $x_j > 0 \Rightarrow u_j = v_j - p_{o_j,d_j,\tau_j}$,

(CS-2) for all $j \in \mathcal{R}$, $u_j > 0 \Rightarrow x_j = 1$,

(CS-3) for all $i \in \mathcal{D}$, $\pi_i > 0 \Rightarrow \sum_{k=1}^{|\mathcal{Z}_i|} y_{i,k} = 1$, (CS-4) for all $i \in \mathcal{D}$ and all $k = 1, \dots, |\mathcal{Z}_i|$, $y_{i,k} > 0 \Rightarrow \pi_i = \sum_{(a,b,t) \in Z_{i,k}} p_{a,b,t}$,

(CS-5) for all $(a, b, t) \in \mathcal{T}$,

$$p_{a,b,t} > 0 \Rightarrow \sum_{j \in \mathcal{R}} x_j \mathbb{1}\{(o_j, d_j, \tau_j) = (a, b, t)\} = \sum_{i \in \mathcal{D}} \sum_{k=1}^{|\mathcal{Z}_i|} y_{i,k} \mathbb{1}\{(a, b, t) \in Z_{i,k}\}.$$

We also provide this following lemma, showing that given any CE outcome, any trip with excessive driver supply has a non-positive price.

**Lemma 5.** Given any plan with anonymous trip prices $(x, \tilde{z}, p)$ that forms a CE, for any $(a, b, t) \in \mathcal{T}$, if there exists a driver $i \in \mathcal{D}$ s.t. $(a, b, t) \in \tilde{z}_i$, then $p_{a,b,t} \leq 0$.

The proof is straightforward. If there exists any trip $(a, b, t)$ with a positive price, and a driver that takes this trip as relocation without a rider, the driver is not getting paid for this trip. This violates driver best response, since getting paid for this trip improves total payment to the driver.

We are now ready to prove Lemma 3.

**Lemma 3** (Welfare Theorem)**.** A dispatching $(x, \tilde{z})$ is welfare-optimal if and only if there exists anonymous trip prices $p$ s.t. the plan $(x, \tilde{z}, p)$ forms a competitive equilibrium. Such optimal CE plans always exist, are efficient to compute, and are individually rational for riders, strictly budget balanced, and envy-free for both riders and drivers.

*Proof.* Given any feasible dispatching $(x, \tilde{z})$, we can construct an integral feasible solution to the LP (10) $(x, y)$ where for all $i \in \mathcal{D}$, $y_{i,k} = 1$ if the movement of driver $i$ in space and time according to the action path $\tilde{z}_i$ under the plan corresponds to $Z_{i,k}$. If $(x, \tilde{z})$ is welfare optimal, $(x, y)$ must be an optimal solution to (10). Also Recall that given any CE plan $(x, \tilde{z}, p)$, Lemma 1 implies that $(x, \tilde{z}, p^+)$ also forms a CE, where $p^+$ is defined s.t. $p^+_{a,b,t} = \max\{p_{a,b,t}, 0\}$. The outline of the two major steps of the proof is as follows:

- Step 1. Given any optimal dispatching $(x, \tilde{z})$, and any optimal solution $(p, \pi, u)$ to the dual LP (16), the CS conditions imply that $\pi$ and $u$ can be interpreted as drivers' total payments and rider's utilities, if the anonymous trip prices is given by $p$. Optimal dual conditions then guarantee driver and rider best responses thus the plan $(x, \tilde{z}, p)$ forms a CE.

- Step 2. Given a CE plan $(x, \tilde{z}, p)$, let $(x, y)$ be the corresponding primal solution, and consider the dual solution $(p^+, \pi, u)$, where $\pi$ and $u$ are driver payments and rider utilities. CS conditions are satisfied between $(x, y)$ and $(p^+, \pi, u)$, thus $(x, \tilde{z}, p)$ is welfare optimal.

This proves the correspondence between CE and optimal plans, and also the existence of CE. The proof of Lemma 2 shows that optimal dispatching can be computed by solving the flow LP. The correspondence between the dual LP and the dual of the flow LP (see Lemma 6 in Appendix B.4) implies that the CE prices can be efficiently computed from solving the dual of the flow LP. Regarding the properties: rider IR and envy-freeness is guaranteed by anonymous trip prices and CE; strict budget balance is guaranteed by the definition of anonymous trip prices; for driver envy-freeness, given any two drivers that start at the same location and time, they have the same set of feasible paths, therefore both get the same highest total payment among those paths.

We now prove the above two steps.

*Step 1: Optimal primal and dual solutions $\Rightarrow$ CE.*

Given an welfare-optimal plan $(x, \tilde{z}, p)$, let $(x, y)$ be the corresponding optimal and integral solution to the primal LP (10), and let $(p, \pi, u)$ be an arbitrary optimal solution to the dual LP (16). We first show that if the anonymous trip prices are given by $p$, then the dual variables $\pi$ and $u$ correspond to drivers' total payment and riders' utilities, respectively. From the complementary slackness conditions, we know:

1. $x_j > 0 \Rightarrow u_j = v_j - p_{o_j,d_j,\tau_j}$ from (CS-1), thus for riders that are picked up: $j \in \mathcal{R}$ s.t. $x_j > 0$, $u_j$ represent the utilities of the rider, which is her value for the trip minus the price for this trip.

2. $u_j > 0 \Rightarrow x_j = 1$ from (CS-2), i.e. in order for a rider to have positive utility, the rider must be picked up. This implies that $x_j = 0 \Rightarrow u_j = 0$, i.e. riders that are not picked up have zero utilities. Combining 1. and 2., we know that $u_j$ correspond to the the rider's utilities.

3. $\pi_i > 0 \Rightarrow \sum_{k=1}^{|\mathcal{Z}_i|} y_{i,k} = 1$ from (CS-3), i.e. in order for a driver to have positive total payment, she must have been dispatched some path. In other words, a driver is not dispatched any path (thus unable to pick up any rider), $\pi_i = 0$ is equal to the driver's total payment.

4. $y_{i,k} > 0 \Rightarrow \pi_i = \sum_{(a,b,t)\in Z_{i,k}} p_{a,b,t}$ from (CS-4), i.e. if driver $i$ takes her $k^{\text{th}}$ feasible path, then $\pi_i$ equals the sum of the prices of each trip covered by this path. We show that this is the driver's total payment by observing that (I) for any rider trip, i.e. $(a, b, t) \in Z_{i,k}$ s.t. $\exists j \in \mathcal{R}$ s.t. $(a, b, t, j) \in \tilde{z}_i$, the driver is paid $p_{a,b,t}$, and (II), for $(a, b, t)$ where the driver relocates without a rider, (CS-5) implies that $p_{a,b,t} = 0$, therefore $p_{a,b,t}$ is also the driver's payment. Combining 3. and 4., we know $\pi_i$ coincides with the the total payment to driver $i$.

We now show that this outcome forms a CE. For rider best response: constraint $u_j \geq 0$ guarantees IR for riders, thus riders that are picked up can afford the price; $v_j - p_{o_j,d_j,\tau_j} > 0 \Rightarrow u_j > 0 \Rightarrow x_j = 1$ implies that all riders that strictly prefer getting pickup up are matched to some driver. For driver best response, the dual constraints (17) and (19) guarantee that for all $i \in \mathcal{D}$, $\pi_i \geq \max_{z_i \in \mathcal{Z}_i} \left\{\sum_{(a,b,t)\in z_i} p_{a,b,t}\right\} = \max_{z_i \in \mathcal{Z}_i} \left\{\sum_{(a,b,t)\in z_i} \max\{p_{a,b,t}, 0\}\right\}$. This completes the proof that the plan with anonymous trip prices $(x, \tilde{z}, p)$ forms a CE.

*Step 2: CE $\Rightarrow$ Optimal primal and dual solutions.*

Let $(x, \tilde{z}, p)$ be a CE plan with anonymous trip prices, and let $(u, \pi)$ be riders' utilities and drivers' total payments under this plan. The plan being feasible implies that corresponding $(x, y)$ is a feasible and integral primal solution. Consider $(p^+, \pi, u)$ as a solution to the dual LP (16)—we show that it's feasible: constraint(19) holds by definition of $p^+$; rider best response implies (18) and (21); driver best response and the non-negativity of prices imply (17) and (20).

We now prove that $(x, y)$ and $(p^+, \pi, u)$ must both be optimal by checking the CS conditions:

1. For (CS-1), given any $j \in \mathcal{R}$ s.t. $x_j > 0$, we know that the rider is picked up, therefore she pays $p_{o_j,d_j,\tau_j}$ and gets utility $u_j = v_j - p_{o_j,d_j,\tau_j}$.

2. For (CS-2), for any rider $j \in \mathcal{R}$, that gets utility $u_j > 0$, she must be picked up since otherwise her utility would be zero, therefore $x_j = 1$ holds.

3. For (CS-3), $\forall i \in \mathcal{D}$, if her total payment $\pi_i > 0$, then it must be the case that she was dispatched to take some path (otherwise her total payment would be zero). Thus $\sum_k y_{i,k} = 1$ must hold.

4. For (CS-4), for each driver $i \in \mathcal{D}$, $y_{i,k} > 0 \Rightarrow y_{i,k} = 1$ implies that driver $i$ takes her $k^{\text{th}}$ feasible path. For each trip $(a, b, t)$ on the path, if the driver picks up a rider, she gets paid $p_{a,b,t}$; if the driver does not pick up a rider, she gets paid zero, which is also equal to $p_{a,b,t}$ (see Lemma 5). Thus her total payment is the sum of the prices of all trips on $Z_{i,k}$, thus $\pi_i = \sum_{(a,b,t)\in Z_{i,k}} p_{a,b,t}$.

5. For (CS-5), $p_{a,b,t} > 0 \Rightarrow \sum_{j\in\mathcal{R},\ (o_j,d_j,\tau_j)=(a,b,t)} x_j = \sum_{i\in\mathcal{D}} \sum_{k=1}^{|Z_i|} y_{i,k}\mathbb{1}\{(a,b,t) \in Z_{i,k}\}$ is implied by Lemma 5— otherwise, there is excessive driver supply for the $(a,b,t)$ trip, implying $p_{a,b,t} = 0$.

This completes the proof of the lemma. □

## B.4 Proof of Lemma 4

### B.4.1 Dual of the Flow LP

The proof of Lemma 4 makes extensive use of the dual of the flow LP, and its correspondence between the dual LP (16), which we establish in this section.

Let $\varphi_{a,t}$ be the dual variable corresponding to the flow balance constraint (37), and let $\mu_j$ be the dual variable corresponding to edge capacity constraint (38). After some simplification, The dual LP of the flow LP (36) can be written as:

$$\min \sum_{i\in\mathcal{D}} \varphi_{\ell_i,\underline{\tau}_i} + \sum_{j\in\mathcal{R}} \mu_j \tag{40}$$

$$\text{s.t. } \varphi_{o_j,\tau_j} - \varphi_{d_j,\tau_j+\delta(o_j,d_j)} + \mu_j \geq v_j, \quad \forall j \in \mathcal{R} \tag{41}$$

$$\varphi_{a,t} - \varphi_{b,t+\delta(a,b)} \geq 0, \quad \forall (a,b,t) \in \mathcal{T} \tag{42}$$

$$\varphi_{a,T} \geq 0, \quad \forall a \in \mathcal{L} \tag{43}$$

$$\mu_j \geq 0, \quad \forall j \in \mathcal{R} \tag{44}$$

Given any feasible solution $(\varphi,\mu)$ of (40), $\varphi_{a,t}$ is usually referred to as the *potential* of the node $(a,t)$, and we call $\varphi$ an *optimal potential* of the MCF problem if there exists $\mu \in \mathbb{R}^{|\mathcal{R}|}$ s.t. $(\varphi,\mu)$ is an optimal solution of (40). The potential $\varphi_{a,t}$ for each node $(a,t)$ can be interpreted as how "useful" it is to have an additional unit of flow originating from this node, and $\mu_j$ for each $j$ can be interpreted as the utility of the rider $j$.

**Complementary Slackness Conditions** Given a feasible solution $f$ to the flow primal LP (36) and a feasible solution $(\varphi,\mu)$ to the flow dual LP (40), both solutions are optimal if and only if the following complementary slackness conditions are satisfied [7].

(CS$_F$-1) for all $j \in \mathcal{R}$, $f(e_j) > 0 \Rightarrow \varphi_{o_j,\tau_j} - \varphi_{d_j,\tau_j+\delta(o_j,d_j)} + \mu_j = v_j$,

(CS$_F$-2) for all $j \in \mathcal{R}$, $\mu_j > 0 \Rightarrow f(e_j) = 1$,

(CS$_F$-3) for all $(a,b,t) \in \mathcal{T}$, $f(((a,t),(b,t+\delta(a,b))) > 0 \Rightarrow \varphi_{a,t} - \varphi_{b,t+\delta(a,b)} = 0$,

(CS$_F$-4) for all $a \in \mathcal{L}$, $f(((a,T),n_0)) > 0 \Rightarrow \varphi_{a,T} = 0$.

The following lemma establishes a one-to-one correspondence between the $\pi$ variables in optimal solutions to the dual LP (16), and the optimal potentials (i.e. the $\varphi$ variables in optimal solutions to (40)) at nodes where the drivers enter the platform.

**Lemma 6.** Given a ridesharing problem satisfying assumptions (S1) and (S2):

(i) Given an optimal solution $(p,\pi,u)$ to the dual LP (16), there exists an optimal solution $(\varphi,\mu)$ to the dual of the flow LP (40) s.t. $\varphi_{\ell_i,\underline{\tau}_i} = \pi_i$ for all $i \in \mathcal{D}$ and that $\mu_j = u_j$ for all $j \in \mathcal{R}$.

(ii) Given an optimal solution $(\varphi,\mu)$ to the dual of the flow LP (40), there exists an optimal solution $(p,\pi,u)$ to the dual LP (16) s.t. $\pi_i = \varphi_{\ell_i,\underline{\tau}_i}$ for all $i \in \mathcal{D}$ and that $u_j = \mu_j$ for all $j \in \mathcal{R}$.

*Proof.* We prove both parts of this lemma by construction.
*Part (i).* We now prove part (i). Given any optimal solution $(p, \pi, u)$ to the dual LP (16), we construct a solution $(\varphi, \mu)$ to (40) as follows, where $\varphi_{a,t}$ represents the highest continuation payoff for any driver from location $a$ and time $t$ onward, and $\mu_j$ represents the utility of rider $j$:

- For all $a \in \mathcal{L}$, let $\varphi_{a,T} = 0$.
- For all $a \in \mathcal{L}$ and all $t < T-1, T-2, \dots, 0$, let

$$\varphi_{a,t} = \max_{b \in \mathcal{L} \text{ s.t. } t+\delta(a,b) \leq T} \left\{ \varphi_{b,t+\delta(a,b)} + p_{a,b,t} \right\}. \tag{45}$$

- For all $j \in \mathcal{R}$, let $\mu_j = \max\{v_j - p'_{o_j,d_j,\tau_j},\ 0\}$, where

$$p'_{a,b,t} \triangleq \varphi_{a,t} - \varphi_{b,t+\delta(a,b)},\ \forall (a,b,t) \in \mathcal{T}. \tag{46}$$

Observe that $p'_{a,b,t} \geq p_{a,b,t}$ holds for all $(a,b,t) \in \mathcal{T}$, since $\varphi_{a,t} \geq \varphi_{b,t+\delta(a,b)} + p_{a,b,t}$ from (45). Moreover, we claim that for any optimal solution $(x, y)$ to the LP (10),

$$p_{a,b,t} = p'_{a,b,t},\ \forall (a,b,t) \in \mathcal{T} \text{ s.t. } \sum_{i \in \mathcal{D}} \sum_{k=1}^{|\mathcal{Z}_i|} y_{i,k} \mathbb{1}\{(a,b,t) \in Z_{i,k}\} > 0, \tag{47}$$

i.e. the prices must coincide for any trips that is covered by at least one driver.

To prove (47), we first observe that for all $(a,t) \in \mathcal{L} \times [T]$, $\varphi_{a,t}$ as defined in (45) is equal to the highest total payment among all possible paths starting from $(a,t)$ to the end of time, given the prices $p$— this is obvious for $t = T$ (since $\varphi_{a,T} = 0$ for al $a$) and also for $t < T$ by induction. Now consider any trip $(a,b,t)$ taken by some driver, say driver 1. Since the outcome forms a CE (since $(x,y)$ and $(p,\pi,u)$ are optimal primal and dual solutions), we know that the total payment to driver 1 from time $t$ onward must be $\varphi_{a,t}$, the highest total payment among all possible paths starting from $(a,t)$. Similarly, the total payment to driver 1 from location $b$ and time $t+\delta(a,b)$ onward is $\varphi_{b,t+\delta(a,b)}$. Since the $(a,b,t)$ trip pays the driver $p_{a,b,t}$, we know that $\varphi_{a,t} = p_{a,b,t} + \varphi_{b,t+\delta(a,b)}$ must hold, which gives us $p'_{a,b,t} = \varphi_{a,t} - \varphi_{b,t+\delta(a,b)} = p_{a,b,t}$.

We are now ready to show that $(\varphi, \mu)$ forms an optimal solution to the dual of the flow LP (40). Constraints (41) to (44) are satisfied by construction, thus $(\varphi, \mu)$ a feasible. What is left to prove is optimality. Let $(x,y)$ be some optimal integral solution to the LP (10). We know that $(x,y)$ and $(p,\pi,u)$ satisfy complementary slackness conditions (CS-1)-(CS-5). We construct an optimal integral solution $f$ to (36) from $(x,y)$ in the same way as in the proof of Lemma 2, and it is sufficient for the optimality to prove that (CS$_F$-1)-(CS$_F$-4) hold between $f$ and $(\varphi, \mu)$:

1. We first show (CS$_F$-1), that $\forall j \in \mathcal{R},\ f(e_j) > 0 \Rightarrow \varphi_{o_j,\tau_j} - \varphi_{d_j,\tau_j+\delta(o_j,d_j)} + \mu_j = v_j$. To show this, first observe that for all $j \in \mathcal{R}$, $f(e_j) > 0 \Rightarrow x_j > 0$ implies that rider $j$ is picked up, thus the trip $(o_j, d_j, \tau_j)$ is taken by some driver, thus $p_{o_j,d_j,\tau_j} = p'_{o_j,d_j,\tau_j}$ by (47). Moreover, $u_j = v_j - p_{o_j,d_j,\tau_j}$ from (CS-1). This gives us: $f(e_j) > 0 \Rightarrow u_j = v_j - p_{o_j,d_j,\tau_j} \Rightarrow \mu_j = v_j - p'_{o_j,d_j,\tau_j} \Rightarrow \varphi_{o_j,\tau_j} - \varphi_{d_j,\tau_j+\delta(o_j,d_j)} + \mu_j = v_j$.
2. To show (CS$_F$-2), that for all $j \in \mathcal{R}$, $\mu_j > 0 \Rightarrow f(e_j) = 1$, note that $p'_{a,b,t} \geq p_{a,b,t}$ for all $(a,b,t) \in \mathcal{T}$. Therefore, for all $j \in \mathcal{R}$, $\mu_j > 0 \Rightarrow v_j - p'_{o_j,d_j,\tau_j} > 0 \Rightarrow v_j - p_{o_j,d_j,\tau_j} > 0 \Rightarrow u_j > 0 \Rightarrow x_j = 1 \Rightarrow f(e_j) = 1$.

3. (CS$_F$-3), which is $\forall e = ((a,t),\ (b, t+\delta(a,b))) \in \mathcal{E}_2,\ f(e) > 0 \Rightarrow \varphi_{a,t} - \varphi_{b,t+\delta(a,b)} = 0$, holds since $f(e) > 0$ implies that there is excessive supply in the dispatching $(x,y)$ for the trip $(a,b,t)$ therefore $p_{a,b,t} = 0$ given (CS-5). (47) then implies that $\varphi_{a,t} - \varphi_{b,t+\delta(a,b)} = p'_{a,b,t} = p_{a,b,t} = 0$.

4. (CS$_F$-4) holds, since $\varphi_{a,T} = 0$ holds for all $a \in \mathcal{L}$ by construction.

*Part (ii).* Let $(\varphi, \mu)$ be an optimal solution to (40). We now construct a solution $(p, \pi, u)$ to (16), where the price $p_{a,b,t}$ is the loss of potential between the origin node $(a,t)$ and the destination node $(b, t+\delta(a,b))$, and the payment $\pi_i$ is equal to the potential where the driver enters:

$$\begin{aligned} u_j &= \mu_j, && \forall j \in \mathcal{R} \\ \pi_i &= \varphi_{\ell_i, \underline{\tau}_i}, && \forall i \in \mathcal{D} \\ p_{a,b,t} &= \varphi_{a,t} - \varphi_{b,t+\delta(a,b)}, && \forall (a,b,t) \in \mathcal{T} \end{aligned}$$

We first show that $(p, \pi, u)$ is a feasible solution to (16).

1. From telescoping sum, for any feasible path $Z_{i,k}$ of driver $i$, starting at $(\ell_i, \underline{\tau}_i)$ and ending at $(a', T)$ for some $a' \in \mathcal{L}$, the total payment on the path is $\sum_{(a,b,t)\in Z_{i,k}} p_{a,b,t} = \varphi_{\ell_i,\underline{\tau}_i} - \varphi_{a',T}$, which is at most $\varphi_{\ell_i,\underline{\tau}_i}$ (since $\varphi_{a',T} \geq 0$ for all $a' \in \mathcal{L}$, guaranteed by (43)). This implies that $\pi_i = \varphi_{\ell_i,\underline{\tau}_i} \geq \sum_{(a,b,t)\in Z_{i,k}} p_{a,b,t}$ for any $k \in \{1, \ldots, |\mathcal{Z}_i|\}$, therefore (17) holds.

2. (41) implies $u_j = \mu_j \geq v_j - (\varphi_{o_j,\tau_j} - \varphi_{d_j,\tau_j+\delta(o_j,d_j)}) = v_j - p_{o_j,d_j,\tau_j}$ thus (18) holds.

3. (42) implies that $p_{a,b,t} \geq 0$ thus (19) holds.

4. (42) and (43) imply that $\varphi_{a,t} \geq 0$ for all $a \in \mathcal{L}$, $t \in [T]$ thus $\pi_i \geq 0$ is satisfied, implying (20).

5. Lastly, (44) implies $u_j \geq 0$, which is (21).

Therefore, $(p, \pi, u)$ is a feasible solution to (16). Regarding the optimality of $(p, \pi, u)$, we know by construction that the objective of (16) is equal to that of (40). Recall the correspondence of optimal solutions that we established in Appendix B.2, that the optimal objective of the flow LP (36) is equal to that of the original LP (10). This implies that the objective of the dual (16) is equal to the objective of the original LP, therefore $(p, \pi, u)$ is an optimal solution of the dual LP (16) such that $\varphi_{\ell_i,\underline{\tau}_i} = \pi_i$ for all $i \in \mathcal{D}$ and that $\mu_j = u_j$ for all $j \in \mathcal{R}$.

This completes the proof of the lemma. □

### B.4.2 Proof of Lemma 4

**Lemma 4** (Lattice Structure)**.** Drivers' total payments $\pi$ among all CE outcomes form a lattice. Moreover, for each driver $i \in \mathcal{D}$, $\Phi_{\ell_i,\underline{\tau}_i}$ and $\Psi_{\ell_i,\underline{\tau}_i}$ are equal to the total payments to driver $i$ in the driver-pessimal and driver-optimal CE plans, respectively.

*Proof.* Step 2 of the proof of Lemma 3 established that the set of possible driver payments among all CE outcomes correspond to the $\pi$ variables among the set of all possible optimal solutions $(p, \pi, u)$ to the dual LP (16). Since Lemma 6 established the correspondence between the $\pi$ variables in (16), and the optimal potentials for the dual of the flow LP (40), what we need to show is that the lattice structure of the optimal potentials, and that $\Phi$ and $\Psi$ reside on the bottom and the top of the lattice. We first prove the lattice structure.

*Step 1. Proof of the Lattice Structure*

Let $(\varphi, \mu)$ and $(\varphi', \mu')$ be two optimal solutions of (40). We prove that the join and the meet of $\varphi$ and $\varphi'$ are both optimal potentials. For all $(a, t) \in \mathcal{L} \times [T]$, let the join and the meet of the potentials be

$$\bar{\varphi}_{a,t} \triangleq \max\left\{\varphi_{a,t},\ \varphi'_{a,t}\right\},$$
$$\underline{\varphi}_{a,t} \triangleq \min\left\{\varphi_{a,t},\ \varphi'_{a,t}\right\}.$$

For convenience of notation, for all $(a, b, t) \in \mathcal{T}$, denote

$$p_{a,b,t} \triangleq \varphi_{a,t} - \varphi_{b,t+\delta(a,b)},$$
$$p'_{a,b,t} \triangleq \varphi'_{a,t} - \varphi'_{b,t+\delta(a,b)},$$

and let $\bar{p}$ and $\underline{p}$ be the potential drop corresponding to the join and the meet:

$$\bar{p}_{a,b,t} \triangleq \bar{\varphi}_{a,t} - \bar{\varphi}_{b,t+\delta(a,b)},$$
$$\underline{p}_{a,b,t} \triangleq \underline{\varphi}_{a,t} - \underline{\varphi}_{b,t+\delta(a,b)}.$$

Finally, for all $j \in \mathcal{R}$, let

$$\bar{\mu}_j \triangleq \max\{v_j - \bar{p}_{o_j,d_j,\tau_j},\ 0\},$$
$$\underline{\mu}_j \triangleq \max\{v_j - \underline{p}_{o_j,d_j,\tau_j},\ 0\}.$$

We first prove that both $(\bar{\varphi}, \bar{\mu})$ and $(\underline{\varphi}, \underline{\mu})$ are feasible solutions to (40). Constraints (41), (43) and (44) hold by construction. For constraint (42), we first show that for all $(a, b, t) \in \mathcal{T}$,

$$\bar{p}_{a,b,t} \in [\min\{p_{a,b,t},\ p'_{a,b,t}\},\ \max\{p_{a,b,t},\ p'_{a,b,t}\}],$$
$$\underline{p}_{a,b,t} \in [\min\{p_{a,b,t},\ p'_{a,b,t}\},\ \max\{p_{a,b,t},\ p'_{a,b,t}\}].$$

We only prove $\bar{p}_{a,b,t} \geq \min\{p_{a,b,t},\ p'_{a,b,t}\}$ here. The proof for the other inequalities are very similar. Assume w.l.o.g. that $\varphi_{a,t} \geq \varphi'_{a,t}$. This implies $\bar{\varphi}_{a,t} = \varphi_{a,t}$. Consider two scenarios: (I) $\varphi_{b,t+\delta(a,b)} \geq \varphi'_{b,t+\delta(a,b)}$ and (II) $\varphi_{b,t+\delta(a,b)} < \varphi'_{b,t+\delta(a,b)}$. For (I), $\bar{\varphi}_{b,t+\delta(a,b)} = \varphi_{b,t+\delta(a,b)}$ thus $\bar{p}_{a,b,t} = \bar{\varphi}_{a,t} - \bar{\varphi}_{b,t+\delta(a,b)} = \varphi_{a,t} - \varphi_{b,t+\delta(a,b)} = p_{a,b,t} \geq \min\{p_{a,b,t},\ p'_{a,b,t}\}$. For (II), we know that $\bar{\varphi}_{b,t+\delta(a,b)} = \max\{\varphi_{b,t+\delta(a,b)},\ \varphi'_{b,t+\delta(a,b)}\} = \varphi'_{b,t+\delta(a,b)}$, thus $\bar{p}_{a,b,t} = \bar{\varphi}_{a,t} - \bar{\varphi}_{b,t+\delta(a,b)} = \varphi_{a,t} - \varphi'_{b,t+\delta(a,b)} \geq \varphi'_{a,t} - \varphi'_{b,t+\delta(a,b)} = p'_{a,b,t} \geq \min\{p_{a,b,t},\ p'_{a,b,t}\}$. Thus we conclude $\bar{p}_{a,b,t} \geq \min\{p_{a,b,t},\ p'_{a,b,t}\}$.

$\varphi$ and $\varphi'$ satisfy (42), which implies $p_{a,b,t} \geq 0$ and $p'_{a,b,t} \geq 0$. Thus $\min\{p_{a,b,t},\ p'_{a,b,t}\} \geq 0$. Therefore $\bar{p}_{a,b,t} \geq 0$ and $\underline{p}_{a,b,t} \geq 0$ both hold. This proves $\bar{\varphi}$ and $\underline{\varphi}$ satisfy (42), and that $(\bar{\varphi}, \bar{\mu})$ and $(\underline{\varphi}, \underline{\mu})$ are both feasible.

Let $f$ be an integral optimal solution to (36) (which is guaranteed to exist). We prove that $(\bar{\varphi}, \bar{\mu})$ and $(\underline{\varphi}, \underline{\mu})$ are both optimal solutions to (40) by showing that the complementary slackness conditions (CS$_F$-1)-(CS$_F$-4) hold between $f$ and $(\bar{\varphi}, \bar{\mu})$, and also between $f$ and $(\underline{\varphi}, \underline{\mu})$. First, observe that the CS (CS$_F$-1)-(CS$_F$-4) hold in between $f$ and $(\varphi, \mu)$ and also between $f$ and $(\varphi', \mu')$.

1. To show (CS$_F$-1), we need to show that for all $j \in \mathcal{R}$ s.t. $f(e_j) > 0$, $\bar{\mu}_j = v_j - \bar{p}_{o_j,d_j,\tau_j}$ and $\underline{\mu}_j = v_j - \underline{p}_{o_j,d_j,\tau_j}$ both hold. (CS$_F$-1) between $f$ and $(\varphi, \mu)$, $(\varphi', \mu')$ imply that if $f(e_j) > 0$, $\mu_j = v_j - p_{o_j,d_j,\tau_j} \geq 0$ and $\mu'_j = v_j - p'_{o_j,d_j,\tau_j} \geq 0$, thus $v_j - \bar{p}_{o_j,d_j,\tau_j} \geq v_j - \max\{p_{o_j,d_j,\tau_j},\ p'_{o_j,d_j,\tau_j}\} = \min\{v_j - p_{o_j,d_j,\tau_j},\ v_j - p'_{o_j,d_j,\tau_j}\} \geq 0$ and $v_j - \underline{p}_{o_j,d_j,\tau_j} \geq v_j - \max\{p_{o_j,d_j,\tau_j},\ p'_{o_j,d_j,\tau_j}\} = \min\{v_j - p_{o_j,d_j,\tau_j},\ v_j - p'_{o_j,d_j,\tau_j}\} \geq 0$. The definitions of $\bar{\mu}_j$ and $\underline{\mu}_j$ then imply $\bar{\mu}_j = v_j - \bar{p}_{o_j,d_j,\tau_j}$ and $\underline{\mu}_j = v_j - \underline{p}_{o_j,d_j,\tau_j}$ must hold.

2. To show (CS$_F$-2), observe that for all $j \in \mathcal{R}$, $\bar{\mu}_j > 0 \Rightarrow v_j - \bar{p}_{o_j,d_j,\tau_j} > 0 \Rightarrow \max\{v_j - p_{o_j,d_j,\tau_j},\ v_j - p'_{o_j,d_j,\tau_j}\} > 0 \Rightarrow \max\{u_j, u'_j\} > 0 \Rightarrow f(e_j) = 1$. Similarly, $\underline{\mu}_j > 0 \Rightarrow f(e_j) = 1$.

3. We now consider (CS$_F$-3). For all $e = ((a,t),\ (b,t+\delta(a,b))) \in \mathcal{E}_2$, $f(e) > 0 \Rightarrow \bar{\varphi}_{a,t} - \bar{\varphi}_{b,t+\delta(a,b)} = \underline{\varphi}_{a,t} - \underline{\varphi}_{b,t+\delta(a,b)} = 0$ holds since $f(e) > 0 \Rightarrow p_{a,b,t} = \varphi_{a,t} - \varphi_{b,t+\delta(a,b)} = 0$ and $p'_{a,b,t} = \varphi'_{a,t} - \varphi'_{b,t+\delta(a,b)} = 0$. Therefore, $0 \le \bar{p}_{a,b,t},\ \underline{p}_{a,b,t} \le 0$ which implies $\bar{\varphi}_{a,t} - \bar{\varphi}_{b,t+\delta(a,b)} = 0$ and $\underline{\varphi}_{a,t} - \underline{\varphi}_{b,t+\delta(a,b)} = 0$.

4. Finally, for (CS$_F$-4): for all $e = ((a,T), n_0) \in \mathcal{E}_3$, $f(e) > 0 \Rightarrow \bar{\varphi}_{a,T} = \underline{\varphi}_{a,T} = 0$ holds since $f(e) > 0 \Rightarrow \varphi_{a,T} = 0$ and $\varphi'_{a,T} = 0$. Thus, $\bar{\varphi}_{a,T} = \underline{\varphi}_{a,T} = 0$.

This completes the proof of the lattice structure of drivers' total payments.

*Step 2. Driver Optimal and Pessimal Plans.*

We now prove the correspondence between the welfare differences and the driver optimal and pessimal payments. $\Psi_{\ell_i,\underline{\tau}_i}$ as defined in (23) denotes the welfare loss from subtracting driver $i$ from the economy. More generally, we can define the welfare loss from losing one driver at a particular location and time as:

$$\Psi_{a,t} \triangleq W(\mathcal{D}) - W(\mathcal{D}\backslash\{(t,T,a)\}), \tag{48}$$

where $W(\mathcal{D}\backslash\{(t,T,a)\})$ is the highest achievable social welfare, if one of the drivers in $\mathcal{D}$ who was supposed to exit the platform at time $T$ now needs to exit the platform at location $a$ at time $t$. Note that this does not specify which particular driver needs to exit, and can be considered as the objective of the flow LP where we simply subtract 1 from the boundary condition $\xi_{a,t}$ at the node $(a,t)$. Also note that this is consistent with (23) since the welfare impact of one driver exiting at $(\ell_i, \underline{\tau}_i)$ is equivalent to that of driver $i$ never entering.

We first show via standard arguments with the residual graph that $\Phi$ and $\Psi$ as we defined in (29) and (48) are optimal potentials for the flow LP. We then show via subgradient arguments that $\Phi$ and $\Psi$ are the bottom and the top of the lattice of the potentials, respectively. Given Lemma 6, and the fact that driver payments among CE outcomes correspond to the optimal solutions of the dual LP (16), we know $\Phi_{\ell_i,\underline{\tau}_i}$ and $\Psi_{\ell_i,\underline{\tau}_i}$ correspond to the bottom and the top of the lattice of driver's total payments among all CE outcomes, hence Lemma 4.

*Step 2.1. $\Phi$ and $\Psi$ are Optimal Potentials:*

Given the MCF problem (32) with graph $G = (\mathcal{N}, \mathcal{E})$ and an optimal integral solution $f$ (which is guaranteed to exist), we first construct the standard residual graph $\tilde{G} = (\mathcal{N}, \tilde{\mathcal{E}})$ where the set of nodes remains the same, and the set of edges $\tilde{\mathcal{E}} = \tilde{\mathcal{E}}_1 \cup \tilde{\mathcal{E}}_2 \cup \tilde{\mathcal{E}}_3$ is consisted of:

- $\tilde{\mathcal{E}}_1 = \{e_j \mid j \in \mathcal{R},\ f(e_j) = 0\} \cup \{\tilde{e}_j \mid j \in \mathcal{R},\ f(e_j) = 1\}$, where $e_j = ((o_j, \tau_j),\ (d_j, \tau_j + \delta(o_j, d_j)))$ is the edge corresponding to rider $j$ with $\gamma(e_j) = -v_j$, $\underline{c}(e_j) = 0$, and $\bar{c}(e_j) = 1$; $\tilde{e}_j = ((d_j, \tau_j + \delta(o_j, d_j)),\ (o_j, \tau_j)))$ is the reversed edge corresponding to rider $j$ s.t. $f(e_j) = 1$, with $\gamma(\tilde{e}_j) = v_j$, $\underline{c}(\tilde{e}_j) = 0$ and $\bar{c}(\tilde{e}_j) = 1$,

- $\tilde{\mathcal{E}}_2 = \mathcal{E}_2 \cup \{\tilde{e} \mid e \in \mathcal{E}_2,\ f(e) > 0\}$, where for all $e = ((a,t),\ (b, t+\delta(a,b))) \in \mathcal{E}_2$, $\tilde{e} = ((b, t + \delta(a,b)),\ (a,t))$ with $\gamma(\tilde{e}) = 0$, $\underline{c}(\tilde{e}) = 0$ and $\bar{c}(\tilde{e}) = f(e)$,

- $\tilde{\mathcal{E}}_3 = \mathcal{E}_3 \cup \{\tilde{e} \mid e \in \mathcal{E}_3\}$ where for all $e = ((a,T),\ n_0) \in \mathcal{E}_3$, $\tilde{e} = (n_0,\ (a,T))$ with $\gamma(\tilde{e}) = 0$, $\underline{c}(\tilde{e}) = 0$ and $\bar{c}(\tilde{e}) = f(e)$.

From the standard argument on the residual graphs [2], we know that the cost of a feasible flow in the residual graph is equal to the incremental cost of the same flow in the original graph. For any node $(a,t)$, the "shortest distance" from this node to the sink $n_0$ refers to the smallest total cost among all paths from $(a,t)$ to $n_0$ in the residual graph. Since the costs are negations of rider values, the shortest distance corresponds to the negation of the maximum incremental welfare created by an additional unit of driver flow starting from $(a,t)$, i.e. $-\Phi_{a,t}$.

For all $(a,b,t) \in \mathcal{T}$, define $p_{a,b,t} \triangleq \Phi_{a,t} - \Phi_{b,t+\delta(a,b)}$, and let $\mu_j \triangleq \max\{v_j - p_{o_j,d_j,\tau_j},\ 0\}$ for all $j \in \mathcal{R}$. We show that $(\Phi, \mu)$ forms an optimal solution to (40). The argument is very similar to that of the reduced cost optimality, however, we include the proof here for completeness. We first show the feasibility of $(\Phi, \mu)$:

1. Constraint (41) holds by definition of $\mu$.

2. For constraint (42), observe that for all $(a,b,t) \in \mathcal{T}$, there exists an edge $((a,t),\ (b,t+\delta(a,b))) \in \tilde{\mathcal{E}}$ with zero cost, thus the shortest distance from $(a,t)$ to $n_0$ is at most zero plus the shortest distance from $(b,t+\delta(a,b))$ to $n_0$, implying $-\Phi_{a,t} \leq -\Phi_{b,t+\delta(a,b)} \Rightarrow \Phi_{a,t} \geq \Phi_{b,t+\delta(a,b)}$,

3. For (43), note that for all $a \in \mathcal{L}$, there exists $((a,T),n_0) \in \tilde{\mathcal{E}}$ with cost $\gamma(e) = 0$. Therefore, the shortest distance $-\Phi_{a,T}$ between $(a,T)$ and $n_0$ is at most 0, i.e. $-\Phi_{a,T} \leq 0 \Leftrightarrow \Phi_{a,T} \geq 0$,

4. (44) holds by definition of $\mu$.

We now show the optimality by examining that the complementary slackness conditions (CS$_F$-1)-(CS$_F$-4) hold between the optimal integral flow $f$ and $(\Phi, \mu)$:

1. For (CS$_F$-1), that $\forall j \in \mathcal{R},\ f(e_j) > 0 \Rightarrow \Phi_{o_j,\tau_j} - \Phi_{d_j,\tau_j+\delta(o_j,d_j)} + \mu_j = v_j$: we know from the definition of $\mu_j$ that to show this, we only need to show that when $f(e_j) > 0$, $v_j - (\Phi_{o_j,\tau_j} - \Phi_{d_j,\tau_j+\delta(o_j,d_j)}) \geq 0 \Leftrightarrow -\Phi_{d_j,\tau_j+\delta(o_j,d_j)} \leq v_j - \Phi_{o_j,\tau_j}$. This holds, since in the residual graph, there is an edge $\tilde{e}_j$ from $(d_j, \tau_j + \delta(o_j,d_j))$ to $(o_j,\tau_j)$ with cost $v_j$, thus the shortest distance from $(d_j, \tau_j + \delta(o_j,d_j))$ to the sink is at most $v_j$ plus the shortest distance from $(o_j,\tau_j)$ to the sink, i.e. $-\Phi_{d_j,\tau_j+\delta(o_j,d_j)} \leq v_j - \Phi_{o_j,\tau_j}$.

2. Now consider (CS$_F$-2): $\forall j \in \mathcal{R},\ \mu_j > 0 \Rightarrow f(e_j) = 1$. To show this, observe that when $\mu_j > 0$, we know $v_j - (\Phi_{o_j,\tau_j} - \Phi_{d_j,\tau_j+\delta(o_j,d_j)}) > 0$, implying $-\Phi_{o_j,\tau_j} > -\Phi_{d_j,\tau_j+\delta(o_j,d_j)} - v_j$, i.e. in the residual graph, the shortest distance from $(o_j,\tau_j)$ to the sink is longer than $-v_j$ plus the shortest distance from $(d_j, \tau_j + \delta(o_j,d_j))$. This means that the edge $e_j = ((o_j,\tau_j),(d_j,\tau_j+\delta(o_j,d_j)))$ with capacity 1 and cost $-v_j$ cannot be present in the residual graph, implying $f(e_j) = 1$.

3. For (CS$_F$-3), that $\forall e = ((a,t),\ (b,t+\delta(a,b))) \in \mathcal{E}_2$ for some $(a,b,t) \in \mathcal{T},\ f(e) > 0 \Rightarrow \Phi_{a,t} - \Phi_{b,t+\delta(a,b)} = 0$: we had already shown that $\Phi_{a,t} - \Phi_{b,t+\delta(a,b)} \geq 0$, thus we only need to show the inequality in the other direction has to hold. This can be proved similarly, after observing that there exists an edge from $(b,t+\delta(a,b))$ to $(a,t)$ in the residual graph with zero cost and non-zero capacity $f(e)$, thus the shortest distance from $(b,t+\delta(a,b))$ to the sink is at most $-\Phi_{a,t}$, implying $\Phi_{a,t} - \Phi_{b,t+\delta(a,b)} \leq 0$,

4. For (CS$_F$-4), that $\forall e = (a,T) \in \mathcal{E}_3$ for some $a \in \mathcal{L},\ f(e) > 0 \Rightarrow \Phi_{a,T} = 0$: assume otherwise, that $\Phi_{a,T} > 0$, meaning that the shortest distance from $(a,T)$ to $n_0$ is negative (i.e. has negative cost)— we can improve the objective of the flow LP by routing one unit of flow that goes form $(a,T)$ to $n_0$ through this alternative shortest path and improve the objective, contradicting the assumption that $f$ is an optimal solution.

This completes the argument that $(\Phi,\mu)$ is an optimal solution to (36). Similarly, we can show that $-\Psi_{a,t}$ corresponds to the shortest distance from the sink $n_0$ to the node $(a,t)$, and that there exists $\mu' \in \mathbb{R}^{|\mathcal{R}|}$ (can be constructed in similar ways as the above $\mu$) s.t. $(\Psi,\mu')$ forms an optimal solution to (36) as well.

*Step 2.2. $\Phi$ and $\Psi$ are the Bottom and Top of the Potential Lattice:*

What is left to show is that $\Phi_{a,t}$ and $\Psi_{a,t}$ must be the bottom and top of the lattice formed by all optimal potentials of (36). For convenience of notation, we now work with the dual of minimization form of the flow LP (32): let $\phi$ be the dual variable corresponding to the flow balance constraint (33) and $\eta$ be the dual variable corresponding to the edge capacity constraint (34), the dual of (32) can be written in the following form:

$$\begin{aligned} \max \quad & \sum_{i\in\mathcal{D}} \phi_{\ell_i,\underline{\tau}_i} + \sum_{j\in\mathcal{R}} \eta_j && (49)\\ \text{s.t.} \quad & \phi_{o_j,\tau_j} - \phi_{d_j,\tau_j+\delta(o_j,d_j)} + \eta_j \le -v_j, && \forall j\in\mathcal{R}\\ & \phi_{a,t} - \phi_{b,t+\delta(a,b)} \le 0, && \forall (a,b,t)\in\mathcal{T}\\ & \phi_{a,T} \le 0, && \forall a\in\mathcal{L}\\ & \eta_j \le 0, && \forall j\in\mathcal{R} \end{aligned}$$

Note that for any optimal solution $(\varphi,\mu)$ to (40), the solution $(\phi,\eta)$ s.t. $\phi=-\varphi$ and $\eta=-\mu$ would be an optimal solution to (49), and vice versa. Therefore the optimal potentials $\phi$ of (49) also forms a lattice, and it is sufficient if we show that $-\Phi_{a,t}$ and $-\Psi_{a,t}$ must be the top and bottom of the lattice formed by all optimal potentials of (49).

Recall that for each $(a,T)\in\mathcal{L}\times[T]$, we defined $\xi_{a,t}=\sum_{i\in\mathcal{D}}\mathbb{1}\{\ell_i=a,\underline{\tau}_i=t\}$, thus fixing the set of riders $\mathcal{R}$, the optimal objective of (32) can be thought of as a function of the $|\mathcal{L}|(T+1)$ by 1 vector $\xi$, denoted as $\omega(\xi)$. It is known that any potential $\phi$ from the set of all optimal solutions of (49) must be a subgradient of the function $\omega(\xi)$ (see the proof of Theorem 5.2 in [8]), but we still include the proof here for completeness. First, $\omega$ is a convex function of $\xi$ (Theorem 5.1 in [8] can easily be generalized to incorporate inequality constraints). Recall that a vector $\phi$ is a *subgradient* of a convex function $\omega$ at $\xi$ if for all $\xi'$,

$$\omega(\xi)+\phi\cdot(\xi'-\xi)\le\omega(\xi').$$

Let $(\phi,\eta)$ be an optimal solution to (49). The strong duality implies $\phi\cdot\xi+\eta\cdot\vec{1}=\omega(\xi)$. Now consider any arbitrary $\xi'$. For any feasible flow $f$ given the boundary condition $\xi'$, weak duality implies $\phi\cdot\xi'+\eta\cdot\vec{1}\le f\cdot\gamma$ where $\gamma$ is the vector of all edge costs. Taking the minimum over all feasible flow $f$, we obtain $\phi\cdot\xi'+\eta\cdot\vec{1}\le\omega(\xi')$. Hence $\phi\cdot\xi'+\eta\cdot\vec{1}-(\phi\cdot\xi+\eta\cdot\vec{1})\le\omega(\xi')-\omega(\xi)\Leftrightarrow\omega(\xi)+\phi(\xi'-\xi)\le\omega(\xi')$, i.e. $\phi$ is a subgradient of $\omega$ at $\xi$.

Now we show that for any subgradient $\phi$ of $\omega$ at $\xi$, the entry $\phi_{a,t}$ is bounded by $-\Psi_{a,t}\le\phi_{a,t}\le-\Phi_{a,t}$.[22] Let $\chi_{a,t}$ be an $|\mathcal{L}|(T+1)$ by 1 vector which takes value 0 except for the $(a,t)$ entry, and $\chi_{a,t}=1$. We know from the definition of subgradient that for all subgradients $\phi$, $\omega(\xi)+\phi\cdot\chi_{a,t}\le\omega(\xi+\chi_{a,t})\Rightarrow\omega(\xi)+\phi_{a,t}\le\omega(\xi+\chi_{a,t})\Rightarrow\phi_{a,t}\le\omega(\xi+\chi_{a,t})-\omega(\xi)=-\Phi_{a,t}$. The last equality holds since the objective $\omega$ is the negation of the optimal total welfare achievable by the vector $\xi$ of driver inflow. Similarly, $\omega(\xi)+\phi\cdot(-\chi_{a,t})\le\omega(\xi-\chi_{a,t})\Rightarrow\omega(\xi)-\phi_{a,t}\le\omega(\xi-\chi_{a,t})\Rightarrow\phi_{a,t}\ge\omega(\xi)-\omega(\xi-\chi_{a,t})=-\Psi_{a,t}$. This implies that $-\Phi$ and $-\Psi$ are the top and the bottom of the lattice formed by the optimal potentials of (32), respectively, and therefore completes the proof of the lemma. $\square$

[22] This is a result of the convexity of $\omega$ and the relationship between directional derivatives and subgradients (see Theorem 3.1.14 in [27]). We include a simple proof here for completeness.

## B.5 Proof of Theorem 2

**Theorem 2.** *The spatio-temporal pricing mechanism is temporally consistent and subgame-perfect incentive compatible. It is also individually rational and strictly budget balanced for any action profile taken by the drivers, and is welfare optimal and envy-free in subgame-perfect equilibrium from any history onward.*

*Proof.* With the same arguments as in the proof of Lemma 4, we know that from any history onward, the plan (re)computed by the STP mechanism forms a CE, and as is pointed out in the outline in the body of the paper, what is left to prove is that a single deviation from the mechanism's dispatches by an individual driver at any time is not useful.

Given any time $t'$ and state $s_{t'}$, let $\Phi_{a,t}^{(t')}(s_{t'})$ be the welfare gain from adding an additional driver at time $t \geq t'$ in location $a$ to the economy starting at time $t'$ and state $s_{t'}$:

$$\Phi_{a,t}^{(t')}(s_{t'}) \triangleq W(\mathcal{D}^{(t')}(s_{t'}) \cup \{(t-t', T, a)\}) - W(\mathcal{D}^{(t')}(s_{t'})). \tag{50}$$

Here, $(t-t', T, a)$ is the type of the additional driver that enters at $a$ at time $t$ in the original economy, and therefore at time $t-t'$ in the time-shifted economy $E^{(t')}(s_{t'})$.

Assume that the current plan $(x^{(t')}(s_{t'}), z^{(t')}(s_{t'}), q^{(t')}(s_{t'}), \pi^{(t')}(s_{t'}))$ is computed at time $t'$, and that no driver had deviated from the plan since time $t'$. Fix any time $t \geq t'$ and let $s_t$ be the state of the platform at time $t$, if all drivers followed the optimal plan computed at time $t'$ up to time $t$. For drivers that are *en route* at time $t$ given $s_t$, there is only one available action (i.e. keep driving) so there is no useful deviation. Consider a driver, say driver $i$, that is available at time $t$ at location $a$ (i.e. $s_{i,t} = (a,t)$). The only available single deviation is to relocate to any location $b \in \mathcal{L}$ that is within reach (i.e. $b \in \mathcal{L}$ s.t. $t + \delta(a,b) \leq T$). We show that this is not useful.

First, if all drivers follow the plan from time $t$ onward, driver $i$'s payment in the remaining time periods would be $\Phi_{a,t}^{(t')}(s_{t'})$, the welfare gain from adding to the economy an additional driver at $(a,t)$, computed at time $t'$. Denote the state of the platform at time $t+1$ as $s_{t+1}$, if all drivers follow the plan at time $t$. Now, at state $s_t$, consider the scenario where the rest of the drivers all follow the plan at time $t$, but driver $i$ deviates and relocates to some location $b \in \mathcal{L}$. Denote the state of the platform at time $t+1$ as $\tilde{s}_{t+1} \triangleq (s_{-i,t+1}, (b, t+\delta(a,b)))$— the states of the rest of the drivers are the same as the case if all drivers follow the plan, and driver $i$ will be available at location $b$ at time $t+\delta(a,b)$. Driver $i$ is not paid at time $t$. The mechanism replans at time $t+1$, and from time $t+1$ onward, driver $i$'s total payment under $\sigma^*$ would be $\Phi_{b,t+\delta(a,b)}^{(t+1)}(\tilde{s}_{t+1})$, the welfare gain from replicating the driver at $(b, t+\delta(a,b))$ computed at time $t+1$ given state $\tilde{s}_{t+1}$. We prove $\Phi_{a,t}^{(t')}(s_{t'}) \geq \Phi_{b,t+\delta(a,b)}^{(t+1)}(\tilde{s}_{t+1})$ by showing:

(i) $\Phi_{a,t}^{(t')}(s_{t'}) \geq \Phi_{a,t}^{(t)}(s_t)$,

(ii) $\Phi_{a,t}^{(t)}(s_t) \geq \Phi_{b,t+\delta(a,b)}^{(t+1)}(s_{t+1})$ for all $b \in \mathcal{L}$ s.t. $t+\delta(a,b) \leq T$, and

(iii) $\Phi_{b,t+\delta(a,b)}^{(t+1)}(s_{t+1}) \geq \Phi_{b,t+\delta(a,b)}^{(t+1)}(\tilde{s}_{t+1})$.

*Part (i):* $\Phi_{a,t}^{(t')}(s_{t'}) \geq \Phi_{a,t}^{(t)}(s_t)$. This is straightforward, since the highest achievable welfare at state $s_{t'}$ with an additional driver at $(a,t)$ is weakly higher than the welfare of the scenario where all drivers follow the plan $(x^{(t')}(s_{t'}), z^{(t')}(s_{t'}), q^{(t')}(s_{t'}), \pi^{(t')}(s_{t'}))$ until time $t$, and then jointly optimize at time $t$ with all the existing drivers (whose states are now $s_t$) and the additional driver at $(a,t)$:

$$W(\mathcal{D}^{(t')}(s_{t'}) \cup \{(t-t', T, a)\}) \geq \left[W(\mathcal{D}^{(t')}(s_{t'})) - W(\mathcal{D}^{(t)}(s_t))\right] + W(\mathcal{D}^{(t)}(s_t) \cup \{(t-t, T, a)\}).$$

Here, $W(\mathcal{D}^{(t')}(s_{t'})) - W(\mathcal{D}^{(t)}(s_t))$ is the welfare achieved by all agents from following the original plan from time $t'$ to time $t-1$; $(t-t, T, a)$ is the type of the additional driver entering at $(a,t)$, in the time-shifted economy starting from $s_t$. The welfare gain is therefore

$$\begin{aligned}
\Phi_{a,t}^{(t')}(s_{t'}) =& W(\mathcal{D}^{(t')}(s_{t'}) \cup \{(t-t', T, a)\}) - W(\mathcal{D}^{(t')}(s_{t'})) \\
\geq& W(\mathcal{D}^{(t')}(s_{t'})) - W(\mathcal{D}^{(t)}(s_t)) + W(\mathcal{D}^{(t)}(s_t) \cup \{(t-t, T, a)\}) - W(\mathcal{D}^{(t')}(s_{t'})) \\
=& W(\mathcal{D}^{(t)}(s_t) \cup \{(t-t, T, a)\}) - W(\mathcal{D}^{(t)}(s_t)) \\
=& \Phi_{a,t}^{(t)}(s_t).
\end{aligned}$$

*Part (ii):* $\Phi_{a,t}^{(t)}(s_t) \geq \Phi_{b,t+\delta(a,b)}^{(t+1)}(s_{t+1})$ *for all* $b \in \mathcal{L}$ *s.t.* $t + \delta(a,b) \leq T$. This is similar to part (i), observing that at state $s_t$, the additional driver at $(a,t)$ can relocate to $b$ (and therefore not contributing to rider welfare at time $t$), while the rest of the drivers follow the original plan at time $t$ and then jointly optimize at time $t+1$:

$$\begin{aligned}
& W(\mathcal{D}^{(t)}(s_t) \cup \{(t-t, T, a)\}) \\
\geq & \left[W(\mathcal{D}^{(t)}(s_t)) - W(\mathcal{D}^{(t+1)}(s_{t+1})) + 0\right] + W(\mathcal{D}^{(t+1)}(s_{t+1}) \cup \{(t + \delta(a,b) - (t+1), T, b)\}).
\end{aligned}$$

Here, $(t + \delta(a,b) - (t+1), T, b) = (\delta(a,b) - 1, T, b)$ is the type of the additional driver entering at time $t + \delta(a,b)$ at location $b$, in the time-shifted economy starting at time $t-1$. This gives us:

$$\begin{aligned}
\Phi_{a,t}^{(t)}(s_t) =& W(\mathcal{D}^{(t)}(s_t) \cup \{(t-t, T, a)\}) - W(\mathcal{D}^{(t)}(s_t)) \\
\geq& W(\mathcal{D}^{(t)}(s_t)) - W(\mathcal{D}^{(t+1)}(s_{t+1})) + W(\mathcal{D}^{(t+1)}(s_{t+1}) \cup \{(\delta(a,b) - 1, T, b)\}) - W(\mathcal{D}^{(t)}(s_t)) \\
=& W(\mathcal{D}^{(t+1)}(s_{t+1}) \cup \{(\delta(a,b) - 1, T, b)\}) - W(\mathcal{D}^{(t+1)}(s_{t+1})) \\
=& \Phi_{b,t+\delta(a,b)}^{(t+1)}(s_{t+1}).
\end{aligned}$$

*Part (iii):* $\Phi_{b,t+\delta(a,b)}^{(t+1)}(s_{t+1}) \geq \Phi_{b,t+\delta(a,b)}^{(t+1)}(\tilde{s}_{t+1})$. First, observe that the only possible difference between $s_{t+1}$ and $\tilde{s}_{t+1}$ is the state of driver $i$. Fixing the state of the rest of the riders as $s_{-i,t+1}$, $\Phi_{b,t+\delta(a,b)}^{(t+1)}(s_{t+1})$ is the welfare gain from adding an additional driver at $(b, t+\delta(a,b))$ where driver $i$ is at $s_{i,t+1}$, whereas $\Phi_{b,t+\delta(a,b)}^{(t+1)}(\tilde{s}_{t+1})$ is the welfare gain from adding an additional driver at $(b, t+\delta(a,b))$ where driver $i$ is also at $(b, t+\delta(a,b))$, the state of driver $i$ that had deviated while the replan happens at time $t+1$.

When $s_{i,t+1} = (b, t+\delta(a,b))$ (i.e. the driver's deviation resulted in the same future state at time $t+1$ as in the scenario that she didn't deviate, e.g. instead of picking up rider $j$ who travels to $d_j$, the driver relocates empty to $d_j$ instead), the inequality trivially holds. When $s_{i,t+1} \neq (b, t+\delta(a,b))$, intuitively, the marginal value of an available driver when there is another available driver at the same location is smaller than the marginal value of an available driver when the existing available driver at some other location, i.e. there is stronger substitution among drivers at the same locations, in comparison to that among drivers at different locations.

More formally, let $\xi^* \in \mathbb{Z}^{|\mathcal{L}|(T+1)}$ be defined as the vector of sources of driver (all drivers but driver $i$) flow given $s_{-i,t+1}$, s.t. for all $a' \in \mathcal{L}$, for all $t' \in [T]$,

$$\begin{aligned}
\xi_{a',t'}^* =& \sum_{i' \neq i} \mathbb{1}\{s_{i',t+1} = (a', t')\} + \sum_{i' \neq i} \mathbb{1}\{s_{i',t+1} = (a'', a', t''),\ t'' + \delta(a'', a') = t'\} \\
&+ \sum_{i' \neq i} \mathbb{1}\{s_{i',t+1} = (o_j, d_j, \tau_j, j),\ d_j = a',\ \tau_j + \delta(o_j, d_j) = t'\},
\end{aligned}$$

i.e. $\xi^*_{a',t'}$ is the number of drivers in $\mathcal{D}\backslash\{i\}$ that are available at $(a',t')$ or are en-route relocating or driving a rider to $(a',t')$.

Moreover, let $(c,t')$ be the location and time where driver $i$ would be available again according to her state $s_{i,t+1}$: if $s_{i,t+1}=(a',t+1)$ for some $a'\in\mathcal{L}$, i.e. driver $i$ is available at time $t+1$, let $c=a'$ and $t'=t+1$. If $s_{i,t+1}=(a'',a',t''+\delta(a'',a'))$ for some $t''\leq t$, i.e. driver $i$ is *en route* relocating to location $a'$, let $c=a'$ and $t'=t''+\delta(a'',a')$. If $s_{i,t+1}=(o_j,d_j,\tau_j,j)$, let $c=d_j$ and $t'=\tau_j+\delta(o_j,d_j)$.

Let $W(\xi)$ be the objective of the flow LP (36) where the driver flow is given by $\xi$, and $\chi_{a,t}$ be the vector of all zeros but a single 1 for the $(a,t)$ entry. The substitution property among drivers at the same location $\Phi^{(t+1)}_{b,t+\delta(a,b)}(s_{t+1})\geq\Phi^{(t+1)}_{b,t+\delta(a,b)}(\tilde{s}_{t+1})$ can now be written as

$$W(\xi^*+\chi_{c,t'}+\chi_{b,t+\delta(a,b)})-W(\xi^*+\chi_{c,t'})\geq W(\xi^*+2\chi_{b,t+\delta(a,b)})-W(\xi^*+\chi_{b,t+\delta(a,b)}).$$

This identity exactly corresponds to the third local exchange property of $M^\natural$ concave functions (equation (4.7) of Theorem 4.1 in [26]), and that the objective function of MCF problems is $M^\natural$ concave (see Example 5 in Section 4.6 of [26]).[23] The objective of the flow problem there is defined as a function of the "sink" nodes in the flow graph, however, the roles of sinks and sources are symmetric: our MCF problem can also be formulated as having a source node at the end of time, where edges go back in time, and the node corresponding to the entering location/time of each driver sinks at most one unit of flow. This completes the proof of the theorem. □

### B.6 Proof of Theorem 4

In this section, we prove the characterization result on driver's total payment, and also discuss drivers' incentives regarding their entrance to the platform, when the actual entering time and location of each driver are not known to the platform.

**Theorem 4** (Uniqueness of STP Payments)**.** *For any ridesharing mechanism that is SPIC, temporally consistent, and space-time invariant, if (i) the downstream plan from any history onward is welfare-optimal and forms a CE, and (ii) drivers do not have incentives to delay entrance to the platform, then the driver payments must be the same as those under the STP mechanism.*

*Proof.* Assume towards a contradiction, that there exists a mechanism with the above properties, and an economy, for which there exists a driver that is paid higher than driver-pessimal payment (i.e. the unit-replica welfare gain) given the plan that the mechanism computes at a certain time and the state of the economy at that time. With temporal-consistency (Definition 10), we can without loss consider only the scenario when the plan is computed at time $t=0$. Denote the economy as $E=(T,\mathcal{L},\delta,\mathcal{D},\mathcal{R})$, and w.l.o.g., we name the driver that is paid higher than driver-pessimal payment driver 1, and assume that her entering location and time are $\ell_1=A\in\mathcal{L}$ and $\underline{\tau}_1\in[T-1]$. Assume that her total payment under time-0 plan is $\pi_1=\Phi_{A,\underline{\tau}_1}+\epsilon$ for some $\epsilon>0$.

Now we construct the following economy $\tilde{E}=(\tilde{T},\tilde{\mathcal{L}},\tilde{\delta},\tilde{\mathcal{D}},\tilde{\mathcal{R}})$ (as shown in Figure 19), with three more locations, one additional time period, and driver 1 replaced by driver 1′ who enters earlier, at location $A''$, and an additional driver entering at the same location and time:

- $\tilde{T}=T+2$, however, we label the time points as $t=-2,-1,0,1,\ldots,T$ for simplicity of notation.

[23]See [26] for a general introduction of $M^\natural$ concavity, and also Chapter 9 of [25] for the related properties of the objectives of network flow problems.

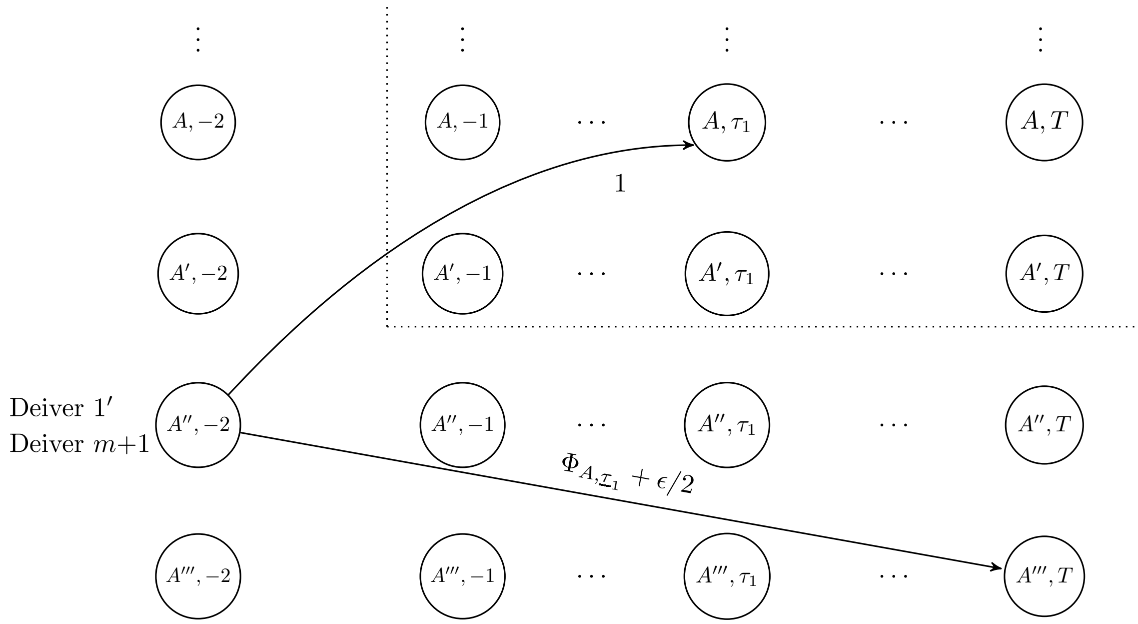



Figure 19: Economy $\tilde{E}$ for the proof of Theorem 4.

- $\tilde{\mathcal{L}} = \mathcal{L} \cup \{A', A'', A'''\}$, where $A', A'', A''' \notin \mathcal{L}$.
- distances $\tilde{\delta} : \mathcal{L} \times \mathcal{L} \to \mathbb{Z}$ such that:
  - $\tilde{\delta}(a,b) = \delta(a,b)$, if $a, b \in \mathcal{L}$,
  - $\tilde{\delta}(A', A') = 1$, $\tilde{\delta}(A, A') = \tilde{\delta}(A', A) = \tau_1 + 1$. For all $a \in \mathcal{L}$ such that $a \neq A$, $\tilde{\delta}(A', a) = \tilde{\delta}(A', A) + \delta(A, a)$ and $\tilde{\delta}(a, A') = \delta(a, A) + \tilde{\delta}(A, A')$,
  - $\tilde{\delta}(A'', A'') = 1$, $\tilde{\delta}(A', A'') = \tilde{\delta}(A'', A') = 1$. For all $a \in \mathcal{L}$, $\tilde{\delta}(A'', a) = \tilde{\delta}(A'', A') + \delta(A', a)$ and $\tilde{\delta}(a, A'') = \delta(a, A') + \tilde{\delta}(A', A'')$,
  - $\tilde{\delta}(A''', A''') = 1$, and for all $a \in \tilde{\mathcal{L}}$ s.t. $a \neq A'''$, $\tilde{\delta}(a, A''') = \tilde{\delta}(A''', a) = T + 2$.
- $\tilde{\mathcal{D}} = \mathcal{D} \backslash \{\theta_1\} \cup \{\theta_1', \ \theta_{m+1}\}$, where driver 1 enters at time $t = -2$, and at location $A''$ instead of $A$: $\theta_1' = (\underline{\tau}_1', \bar{\tau}_1', \ell_1') = (-2, T, A'')$; and we add an additional driver at the same location and time: $\theta_{m+1} = (-2, T, A'')$.
- $\tilde{\mathcal{R}} = \mathcal{R} \cup \{(A'', A, -2, 1), \ (A'', A''', -1, \Phi_{A,\underline{\tau}_1} + \epsilon/2)\}$, meaning that we add two additional riders, as shown in Figure 19.

First, we can check that the distances $\tilde{\delta}$ defined as above satisfies the triangle inequality. If $a, b, c \in \mathcal{L}$, then $\tilde{\delta}(a,c) \leq \tilde{\delta}(a,b) + \tilde{\delta}(b,c)$ holds from the assumption that $\delta$ satisfies triangle inequality. Note that $a$ and $c$ are symmetric, we consider when $a = A'$ and when $b = A'$: when $a = A'$, and when $b, c \in \mathcal{L}$, $\tilde{\delta}(a,c) = \tilde{\delta}(A', c) = \tilde{\delta}(A', A) + \delta(A, c) \leq \tilde{\delta}(A', A) + \delta(A, b) + \delta(b, c) = \tilde{\delta}(A', b) + \tilde{\delta}(b, c)$; when $b = A'$ and when $a, c \in \mathcal{L}$, $\tilde{\delta}(a, b) + \tilde{\delta}(b, c) = \tilde{\delta}(a, A') + \tilde{\delta}(A', c) = \delta(a, A) + \tilde{\delta}(A, A') + \tilde{\delta}(A', A) + \delta(A, c) \geq \delta(a, c) + \tilde{\delta}(A, A') + \tilde{\delta}(A', A) > \delta(a, c)$. The same argument can be repeated for the case when $a = A''$ and when $b = A''$, and it is also easy to see that if any of $a, b, c$ is $A''$, the triangle inequality holds.

We now claim that under the welfare optimal plan, at time $t = -2$, one of the two drivers at $(A'', -2)$ picks up the rider going from $A''$ to $A$ who is willing to pay 1, and the other driver is dispatched to pick up the rider going to $A'''$ who is willing to pay $\Phi_{A,\underline{\tau}_1} + \epsilon/2$. First, we can check that $\tilde{\delta}(A'', a) = \delta(A'', A) + \delta(A, a)$ for all $a \in \mathcal{L}$, $a \neq A$, therefore moving to any location $a \neq A$ in $\mathcal{L}$ is no better than moving to $A$ first, and then to $a$ from $A$. Thus the only decision to make for drivers $1'$ and $m+1$ is whether to move to $A$ or to stay in $A'$, $A''$ or $A''$. Given that there is only one rider trip $(A'', A''', -1, \Phi_{A,\underline{\tau}_1} + \epsilon/2)$ with origin and destination that are not in $\mathcal{L}$, under at least one of the welfare-optimal dispatching, one of the two drivers should move to $A$ at time $-2$. What is left to show is that the other driver should move to $A'''$: this is obvious, since the welfare gain from picking the rider going to $A'''$ is $\Phi_{A,\tau_1} + \epsilon/2$, in comparison to $\Phi_{A,\tau_1}$, the value of an additional driver at $A$.

Given the optimal dispatching, we w.l.o.g. assume that at time $-2$, driver $m+1$ is dispatched to go to $A'''$, and that driver $1'$ is dispatched to go to $A$. We know from IR and the requirement that the mechanism uses anonymous trip prices, that the total payment to driver $m+1$, which is equal to the anonymous trip prices for the $(A'', A''', -2)$ trip, cannot exceed $\Phi_{A,\underline{\tau}_1} + \epsilon/2$. From envy-freeness on the driver's side, the total payment to driver $1'$ cannot exceed $\Phi_{A,\underline{\tau}_1} + \epsilon/2$ either.

We now show a useful deviation of agent $1'$ at time $t = -2$: by declining the dispatch to pick up the rider $(A'', A, -2, 1)$, and drive to location $A'$, the driver triggers a replanning at time $t = -1$. While updating the plan, the state of driver $1'$ is $(A', -1)$. Driver $m+1$ is *en route* to location $A'''$ and is no longer able to complete any trip so is no longer relevant. Similarly, locations $A''$ and $A'''$ are no longer relevant since there is no more trip to or from these locations from time $-1$ onward. Therefore, what is relevant for a space-time consistent mechanism is the locations and times within the area within the dotted boundaries. In comparison to the original economy, the only difference is that there is one additional time $t = -1$, and that driver $1'$ enters earlier than the original driver 1. From time-invariance and space-time consistency, we know that for the economy that starts at time $-1$, if the driver now reports $(A, \underline{\tau}_1)$ as her entering location and time and actually enter at $(A, \underline{\tau}_1)$, her payment would be $\Phi_{A,\underline{\tau}_1} + \epsilon$. From the no delayed entrance condition, we know that driver $1'$ at $(A', -1)$ must also be paid at least $\Phi_{A,\underline{\tau}_1} + \epsilon$. This is a useful deviation. $\square$

### B.6.1 Truthful Reporting of Driver Entrance

Throughout the paper, we assumed a complete information model, where the mechanism knows about the entering location and time for all the drivers. In this section, we discuss the scenario where these are drivers' private information, and the mechanism asks the drivers to report their entrance information at the beginning of the planning horizon, then compute and update the plans accordingly. Here, we still assume that all drivers stay until at least the end of the planning horizon.

**Theorem 5.** *Under the STP mechanism, for driver $i$ with type $(\underline{\tau}_i, T, \ell_i)$, it is not useful for her to report $(\hat{\underline{\tau}}_i, T, \hat{\ell}_i) \in \mathcal{L} \times [T]$ where $\hat{\underline{\tau}}_i \geq \underline{\tau}_i + \delta(\ell_i, \hat{\ell}_i)$, and then enter the platform $(\hat{\ell}_i, \hat{\underline{\tau}}_i)$.*

*Proof.* First, for driver $i$ whose true entering location and time is $(\ell_i, \underline{\tau}_i)$, the driver is only able to enter at $(\ell_i, \underline{\tau}_i)$, or at any $(\hat{\ell}_i, \hat{\underline{\tau}}_i) \in \mathcal{L} \times [T]$ s.t. $\hat{\underline{\tau}}_i \geq \underline{\tau}_i + \delta(\ell_i, \hat{\ell}_i)$. Assume that drivers all follow the dispatches once they have entered the platform, which is SPIC. If driver $i$ reports truthfully, her total payment would be $\Phi_{\ell_i, \underline{\tau}_i}$, the welfare gain from an additional driver added at $(\ell_i, \underline{\tau}_i)$ to the original economy.

Following the same notation as in the proof of Theorem 2, we denote the boundary condition for the flow problem of the economy except for agent $i$ as $\xi$, i.e. for any $(a, t) \in \mathcal{L} \times [T]$, let $\xi_{a,t} = \sum_{i' \neq i} \mathbb{1}\{\underline{\tau}_{i'} = t, \ell_{i'} = a\}$. Let $W(\cdot)$ be the optimal objective of the corresponding flow problem, $\Phi_{\ell_i, \underline{\tau}_i}$ can be written as:

$$\Phi_{\ell_i, \underline{\tau}_i} = W(\xi + 2\chi_{\ell_i, \underline{\tau}_i}) - W(\xi + \chi_{\ell_i, \underline{\tau}_i}).$$

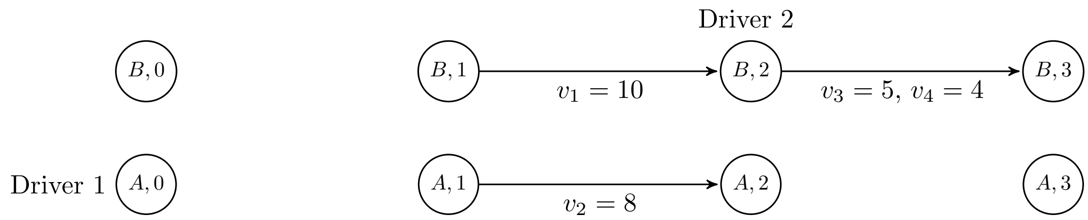



Figure 20: The economy in Example 4, with two locations $A$, $B$, two time periods and four riders.

Here, $\chi_{\ell_i,\underline{\tau}_i}$ is a $|\mathcal{L}|$ by $T+1$ vector with all zero entries, except a single 1 at the $(\ell_i, \underline{\tau}_i)$ entry. If the driver reports $(\hat{\ell}_i, \hat{\underline{\tau}}_i)$ as her entering location and time, and actually enters at $(\hat{\ell}_i, \hat{\underline{\tau}}_i)$, her equilibrium payoff for the rest of the planning horizon can be written as

$$W(\xi + 2\chi_{\hat{\ell}_i,\hat{\underline{\tau}}_i}) - W(\xi + \chi_{\hat{\ell}_i,\hat{\underline{\tau}}_i}).$$

This is not a useful deviation, since

$$\begin{aligned} &W(\xi + 2\chi_{\ell_i,\underline{\tau}_i}) - W(\xi + \chi_{\ell_i,\underline{\tau}_i}) - (W(\xi + 2\chi_{\hat{\ell}_i,\hat{\underline{\tau}}_i}) - W(\xi + \chi_{\ell_i,\underline{\tau}_i})) \\ \geq &W(\xi + 2\chi_{\ell_i,\underline{\tau}_i}) - W(\xi + \chi_{\ell_i,\underline{\tau}_i}) - (W(\xi + \chi_{\ell_i,\underline{\tau}_i} + \chi_{\hat{\ell}_i,\hat{\underline{\tau}}_i}) - W(\xi + \chi_{\ell_i,\underline{\tau}_i})) \\ = &W(\xi + 2\chi_{\ell_i,\underline{\tau}_i}) - W(\xi + \chi_{\ell_i,\underline{\tau}_i} + \chi_{\hat{\ell}_i,\hat{\underline{\tau}}_i}) \geq 0. \end{aligned}$$

The first inequality holds due to the local exchange properties of the $M^\natural$ concave functions, and the last inequality holds since a driver that enters the platform at $(\ell_i, \underline{\tau}_i)$ is at least as useful as a driver that enters later in time at a location that's reachable from $(\ell_i, \underline{\tau}_i)$. $\square$

This result on the truthfulness of driver reports, however, consider only the scenario that the driver actually enters at the location and time as she had reported. We may also consider a mechanism that takes the drivers' reports of entering location and time, plans accordingly at the beginning of the planning horizon, but replans if any driver's entrance is different from expected, without penalizing any driver that had deviated. The following example shows that when allowing arbitrary driver entrance regardless of their report, the STP mechanism does not incentivize the drivers to truthfully report their entering location and time. Moreover, no mechanism with the set of economic properties of the STP mechanism can be truthful.

**Example 4.** Consider the economy as shown in Figure 20. The planning horizon is $T = 3$ and there are two locations $\mathcal{L} = \{A, B\}$ with unit distances $\delta(a, b) = 1$ for all $a, b \in \mathcal{L}$. There is one driver entering at time $\underline{\tau}_1 = 0$ at location $\ell_1 = A$ and leaves at time $\bar{\tau}_1 = 3$. There is another driver, whose true entering time and location is $\underline{\tau}_2 = 2$ and $\ell_2 = B$. There are four riders:

- Rider 1: $o_1 = B$, $d_1 = B$, $\tau_1 = 1$, $v_1 = 10$,
- Rider 2: $o_2 = A$, $d_2 = A$, $\tau_2 = 1$, $v_2 = 8$,
- Rider 3: $o_3 = B$, $d_3 = B$, $\tau_3 = 2$, $v_3 = 5$,
- Rider 4: $o_4 = B$, $d_4 = B$, $\tau_4 = 2$, $v_4 = 4$.

Under the STP mechanism, if both drivers report their entrance location and time truthfully, the welfare-optimal plan dispatches driver 1 to take the path $((A, B, 0),\ (B, B, 1),\ (B, B, 2))$ and pick up riders 1 and 3. Driver 2 is dispatched to pick up rider 4, and her payment would be 0, the welfare gain from an additional driver entering at $(B, 2)$.

However, if driver 2 reports $(B, 1)$ as her entering time and location, then under the STP mechanism, driver 1 would be dispatched to go to $(A, 1)$ to pick up rider 2 at time 1. When time

1 comes, driver 2 fails to enter at $(B, 1)$, and regardless of any future entrance of driver 2, the optimal plan at time 1 is that driver 1 pick up rider 2. When time 2 comes, driver 2 can then decide to actually enters location $B$. The mechanism would replan again, dispatching driver 2 to pick up rider 3. The welfare gain from an additional driver at $(B, 2)$ would be 4, thus the driver's new payment would be 4, and this is a useful deviation.

Now consider any mechanism with the STP properties, as required in the statement of Theorem 4. If driver 2 reports truthfully, Theorem 4 requires that the payment to her cannot exceed her marginal welfare gain, which is zero. However, if she reports $(B, 1)$ as her entering location and time, and actually enter at $(B, 2)$, optimal dispatching and envy-freeness requires that she gets paid at least 4. This is a useful deviation, and proves the impossibility in incentivizing truthful driver reports on entrance information. □

# C Additional Examples and Discussions

We provide in this section additional examples and discussions omitted from the body of the paper.

## C.1 The Driver-Optimal Mechanism

A natural variation on the STP mechanism is to consider the driver-optimal analogue, which always computes a driver-optimal competitive equilibrium plan at the beginning of the planning horizon, or upon deviation of any driver. This mechanism pays each driver the externality she brings to the economy, and corresponds to the reasoning of the VCG mechanism. The following example shows, however, that the driver-optimal mechanism is not incentive compatible. This is because as time progresses, the set of paths that are available to the drivers shrink, thus the welfare loss from losing some driver may increase. Because of this, it may sometimes be profitable for such drivers to deviate, trigger the replanning and get higher total payments in subsequent time periods.

**Example 5.** Consider the economy illustrated in Figure 21 with three locations, three time periods and symmetric distances $\delta(A, A) = \delta(B, B) = \delta(C, C) = \delta(B, C) = 1$ and $\delta(A, B) = \delta(A, C) = 2$. Two drivers enter the platform at time 0 at location $B$, and three riders have types:

- Rider 1: $o_1 = C$, $d_1 = C$, $\tau_1 = 1$, $v_1 = 1$,
- Rider 2: $o_2 = C$, $d_2 = C$, $\tau_2 = 2$, $v_2 = 5$,
- Rider 3: $o_3 = A$, $d_3 = A$, $\tau_3 = 1$, $v_3 = 1$.

In a welfare-optimal dispatching as shown in Figure 21, driver 1 is dispatched to take the path $z_1 = ((B, C, 0),\ (C, C, 1),\ (C, C, 2))$ and to pick up riders 1 and 2. Driver 2 takes the path $z_2 = ((B, A, 0),\ (A, A, 2))$ and picks up rider 3. One driver-optimal CE plan sets anonymous trip prices $p_{C,C,1} = 0$ and $p_{C,C,2} = p_{A,A,2} = 1$, so that each driver is paid the welfare-loss of 1, if she was eliminated from the economy.

Assume that driver 2 follows the mechanism and starts to drive toward $A$ at time 0, we show a useful deviation of driver 1 by rejecting the dispatched relocation to $C$ and staying in location $B$ to trigger a replanning at time 1. At time 1, driver 2 is already *en route* to $A$ thus is only able to pick up rider 3 at time 2. Driver 1 would be asked to relocate to $C$ and pick up rider 2. The price $p_{C,C,2}$ in the driver-optimal CE plan would actually be 5, the welfare loss if the economy loses driver 1. This is higher than driver 1's payment from following the dispatches at all times. □

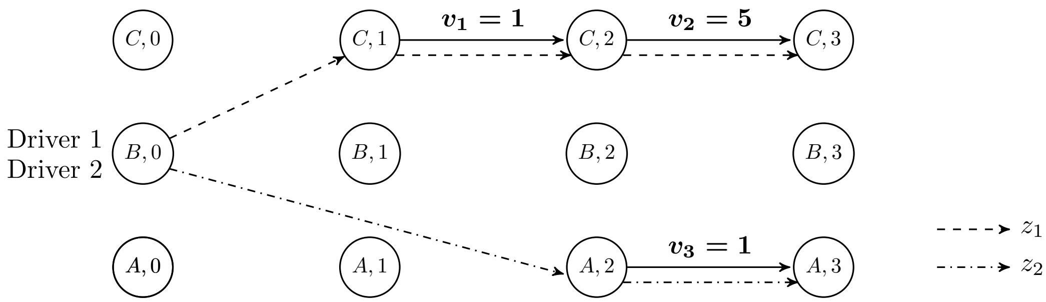



Figure 21: Illustration of the toy economy II in Example 5 with three locations $A$, $B$, $C$, three time periods, two drivers starting at $(B, 0)$ and three riders with values 1, 5 and 1, respectively. Under a welfare optimal plan, driver 1 picks up riders 1 and 2 and driver 2 picks up rider 3.

A variation on the driver-optimal mechanism where drivers' payments are shifted in time is equivalent to the *dynamic VCG mechanism* [6, 12]. The dynamic VCG mechanism is not incentive compatible because the existence of a driver at some certain time may exert negative externality on the economy, in which case the payment to the driver would be negative for that time period. The driver would then have incentives to decline the dispatch and avoid such payment. See the discussions and examples in Appendix D.1.

## C.2 LP Integrality and Existence of CE

We show via the following two examples that when either of the assumptions (S1) and (S2) is violated, the LP relaxation (10) of the ILP (5) may no longer be integral, and that welfare-optimal competitive equilibrium outcomes as defined in Definition 3 may not exist. We first examine the case where drivers may have different times of exiting the platform.

**Example 6** (Different Driver Exit Times)**.** Consider the economy as shown in Figure 22 with three locations $\mathcal{L} = \{A, B, C\}$ and three time periods. The distances are symmetric and given by $\delta(A, A) = \delta(B, B) = \delta(C, C) = \delta(A, B) = \delta(B, C) = 1$, and $\delta(A, C) = 2$. There are three drivers, entering and exiting at:

- $\ell_1 = A$, $\underline{\tau}_1 = 0$, $\bar{\tau}_1 = 3$,
- $\ell_2 = B$, $\underline{\tau}_2 = 0$, $\bar{\tau}_2 = 2$,
- $\ell_3 = B$, $\underline{\tau}_3 = 1$, $\bar{\tau}_3 = 3$,

and there are six riders with types:

- $o_1 = A$, $d_1 = C$, $\tau_1 = 0$, $v_1 = 5$,
- $o_2 = A$, $d_2 = B$, $\tau_2 = 1$, $v_2 = 7$,
- $o_3 = A$, $d_3 = B$, $\tau_3 = 1$, $v_3 = 1$,
- $o_4 = B$, $d_4 = A$, $\tau_4 = 1$, $v_4 = 2$,
- $o_5 = B$, $d_5 = A$, $\tau_5 = 1$, $v_5 = 5$,
- $o_6 = B$, $d_6 = A$, $\tau_6 = 2$, $v_6 = 4$.

In the unique optimal integral solution, Driver 1 takes the path $z_1^* = ((A, A, 0),\ (A, B, 1),\ (B, A, 2))$ and picks up riders 2 and 6. Driver 2 takes the path $z_2^* = ((B, B, 0),\ (B, A, 1))$ and picks up rider 4. Driver 3 takes the path $z_3^* = ((B, A, 1),\ (A, A, 2))$ and picks up rider 5. The total social welfare is $v_2 + v_6 + v_4 + v_5 = 18$. The optimal solution of the LP, however, is not integral. Each driver $i$ takes each of their two paths $z_i$ and $z_i'$ with probably 0.5:

- $z_1 = ((A, C, 0),\ (C, C, 2))$, $z_1' = ((A, A, 0),\ (A, B, 1),\ (B, A, 2))$,

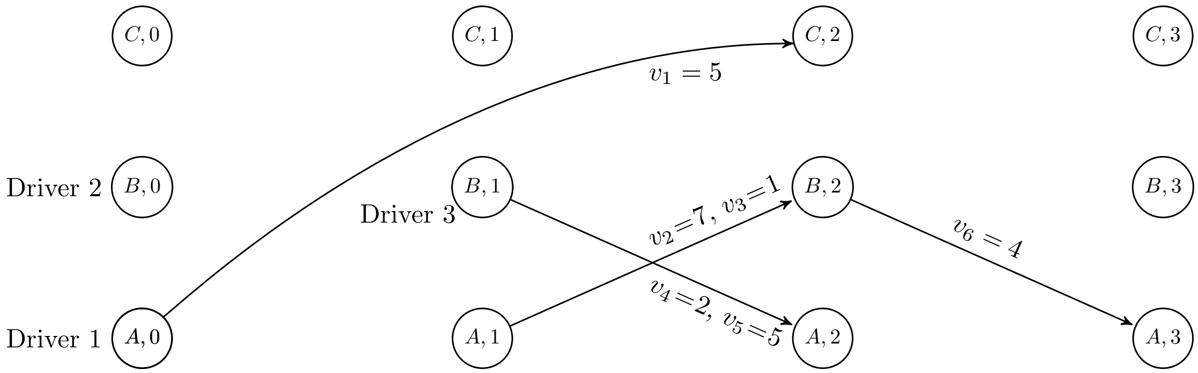



Figure 22: The economy in Example 6 with three locations $A$, $B$, $C$, three time periods, 6 riders, three drivers starting at $(A, 0)$, $(B, 0)$ and $(B, 1)$, where driver 2 exits the platform at time 2.

- $z_2 = ((B, B, 0),\ (B, A, 1))$, $z_2' = ((B, A, 0),\ (A, B, 1))$,
- $z_3 = ((B, A, 1),\ (A, A, 2))$, $z_3' = ((B, B, 1),\ (B, A, 2))$.

The riders 2, 5 and 6 are picked up with probability 1, whereas rider 1 is picked up with probability 0.5. The total social welfare is $0.5v_1 + v_2 + v_5 + v_6 = 18.5 > 18$. There is a unique solution to the dual LP (16), which implies anonymous trip prices of $p_{A,C,0} = 5$ and $p_{A,B,1} = p_{B,A,1} = p_{B,A,2} = 2.5$. These prices do not support the optimal integral solution, since rider 4 is willing to pay only $v_2 = 2$ but is picked up and charged 2.5.

Moreover, we show that no anonymous origin-destination prices support the optimal integral dispatch in competitive equilibrium.[24] First, rider 1 with value 5 is not picked up, therefore the price for the $(A, C, 0)$ trip needs to be at least $p_{A,C,0} \geq 5$. In order for driver 1 to not regret not taking this trip, the prices for her trips need to be at least $p_{A,B,1} + p_{B,A,2} \geq 5$.

Since rider 4 with value 2 is picked up, the price for trip $(B, A, 1)$ can be at most 2. As a consequence, the price for the trip $(A, B, 1)$ cannot exceed 2 either, since otherwise, driver 2 would have incentive to take the path $((B, A, 0),\ (A, B, 1))$ instead. This implies that the price for the trip $(B, A, 2)$ needs to be at least 3. Note that driver 3 now prefers taking the path $((B, B, 1),\ (B, A, 2))$ and get paid at least 3, in comparison to the dispatched trip $((B, A, 1),\ (A, A, 2))$ and gets paid at most 2. This shows that no anonymous OD price supports the welfare-optimal outcome in competitive equilibrium. $\square$

The reason integrality failed is that the ridesharing problem can no longer be reduced to an MCF problem in the way that we discuss in Appendix B.2.1 without loss of generality. In the standard MCF problem, there is a single type of flow flowing through the network, and the optimal flow is guaranteed to be integral. When drivers have different exiting times, if all units of flow are still treated as homogeneous, the resulting decomposed flow may not send the correct drivers to leave at the correct times. As an example, the optimal homogeneous flow with the same boundary condition in this example can be decomposed into the following three paths: $((A, C, 0))$, $((B, A, 0), (A, B, 1), (B, A, 2)), ((B, A, 1), (A, A, 2))$ with a total social welfare of 21. However, it cannot be implemented since it is driver 2 who enters at $(B, 0)$ and needs to exit at time 2, but in this decomposition, the flow that corresponds to driver 1 exits at time 2.

[24] In general, the non-existence of CE does not imply that there do not exist dynamic ridesharing mechanisms that are SPIC, since a mechanism determining a CE plan is not necessary for the mechanism to be incentive compatible.

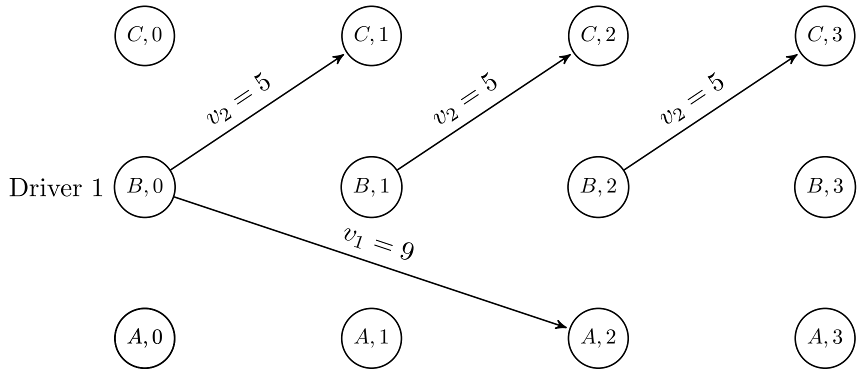



Figure 23: The economy in Example 7 with three locations $A$, $B$, $C$, three time periods, 2 riders, and one driver entering the platform at $(B, 0)$.

When drivers have different exiting times, the MCF problem has heterogeneous flows. Similarly, we can construct examples showing that the optimal solutions to the ILP and LP do not coincide if drivers have preference over which location to they end up with at the end of the planning horizon, unless all drivers start at the same time and location.

The following example examines the case where riders are patient and may be willing to wait.

**Example 7** (Patient Riders)**.** Consider the economy as illustrated in Figure 23 with three locations $\mathcal{L} = \{A, B, C\}$ and three time periods. The pairwise distances are symmetric and given by $\delta(A, A) = \delta(B, B) = \delta(C, C) = \delta(B, C) = 1$ whereas $\delta(A, B) = \delta(A, C) = 2$. There is a single driver entering at location $B$ at time 0 who would stay until the end of the planning horizon.

There are two riders. Rider 1 is impatient, requesting a trip at time 0 from $B$ to $A$, and is willing to pay $v_1 = 9$. Rider 2 is willing to pay $v_2 = 5$ for a trip from $B$ to $C$ at time 0, but is willing to wait for at most two time periods.

In the optimal integral solution, driver 1 takes the path $((B, A, 0),\ (A, A, 2))$, picks up rider 1, and achieves total social welfare of $v_1 = 9$. In the optimal solution to the LP (10), however, the driver takes the paths $((B, A, 0),\ (A, A, 2))$ and $((B, C, 0),\ (C, B, 1),\ (B, C, 2))$, each with probability 0.5. The path $((B, C, 0),\ (C, B, 1),\ (B, C, 2))$ seemingly have a total value of 10, therefore the objective of the LP would be $10 \times 0.5 + 9 \times 0.5 = 9.5 > 9$.

The optimal integral solution is not supported by anonymous OD prices in CE either— since rider 2 is not picked up, the prices for the $(BC)$ trips starting at times 0, 1 and 2 need to be at least 5. Thus the total payment for the path $((B, C, 0),\ (C, B, 1),\ (B, C, 2))$ is at least 10, however, the driver's dispatched trip $(B, A, 0)$ pays at most $v_2 = 9$. $\square$

The reason that integrality fails is that unlike the setting when riders are impatient, there is no direct way of reducing the problem to an MCF problem without loss. More specifically, in the MCF problem, each rider corresponds to a single edge in the flow graph with edge cost equal to the negation of the rider's value. If the rider is patient, there would be multiple edges that may correspond to the same rider, and there is no easy way expressing the constraint in the MCF problem that a rider cannot be picked up more than once without breaking the integrality of the MCF problem.

## C.3 Rider Incentives

The following example illustrates (i) the trip-prices and rider-utilities under all CE outcomes may not have lattice structure and (ii) the rider-side VCG prices do not coincide with the prices in the driver-pessimal CE plan, and (iii) no welfare-optimal CE mechanism, including the STP mechanism, incentivizes riders to truthfully report their values.

**Example 8** (Example 1 Continued)**.** Consider the economy in Example 1. Under the welfare-optimal dispatching, the driver takes the path $((A, A, 0), (A, A, 1))$ and picks up riders 1 and 2, achieving a social welfare of $v_1 + v_2 = 11$. In the driver pessimal CE plan, the prices for the trips would be $p_{A,B,0} = 8$, and $p_{A,A,0} + p_{A,A,1} = 8$, therefore $p_{A,A,0} = 5$, $p_{A,A,1} = 3$ and $p_{A,A,0} = 2$ and $p_{A,A,1} = 6$ would both support the driver-pessimal CE outcome.

*Lattice Structure:* We can check that the lowest price for the trips $(A, A, 0)$ and $(A, A, 1)$ under all CE outcomes would be 2 and 3 respectively. However, setting $p_{A,A,0} = 2$ and $p_{A,A,1} = 3$ wouldn't form a CE, since rider 3 is willing to pay 8 thus $p_{A,B,0} \geq 8$. This implies that trip prices under all CE outcomes do not form a lattice. As a consequence, riders' utilities under all CE outcomes do not form a lattice either.

*Rider-side VCG Prices:* Moreover, we can check that $p_{A,A,0} = 2$ is what rider 1 should be charged under the VCG payment rule: if rider 1 isn't present, rider 3 gets picked up thus the total welfare for the rest of the economy increases from $v_2 = 6$ to $v_3 = 8$. Similarly, rider 2's VCG payment would be $p_{A,A,1} = 3$. This shows that the VCG payment on the rider side may not support a welfare-optimal CE outcome.

*Rider-side IC:* This example also implies that the STP mechanism is not incentive compatible on the rider's side. Under any driver-pessimal outcome, we know that one of the riders 1 and 2 would be charged a payment that is higher than their VCG price. A simple analysis would show that if the rider who is charged higher than the VCG price reports the VCG price as her value, then her payment under the STP mechanism would be exactly her VCG price. This is a useful deviation. More generally, this shows that no welfare-optimal CE outcome would be incentive compatible on the rider's side, since $p_{A,A,0} + p_{A,A,1} \geq 8$ under any CE outcome. □

It is not a coincidence that the lowest possible prices for each rider under all CE outcomes is equal to their rider-side VCG prices. The following theorem shows that the minimum CE prices and the rider-side VCG prices always coincide.

**Theorem 6** (Minimum CE = Rider-Side VCG)**.** *For a rider that is picked up in any welfare-optimal dispatching, her rider-side VCG price is equal to the minimum price for her trip among all CE outcomes.*

*Proof.* For simplicity of notation, assume that driver $j \in \mathcal{R}$ requests the trip $(a, b, t)$, has value $v_j$, and is picked up under some welfare-optimal dispatching. We are going to prove:

(i) the price $p_{a,b,t}$ under any CE outcome is at least the rider-side VCG payment for rider $j$, and

(ii) there exists an CE outcome where rider $j$'s trip price is at most her rider-side VCG payment.

Combining (i) and (ii), we know that the rider-side VCG payment has to be the lowest CE price for the trip among all CE outcomes.

Let $\tilde{W}(\mathcal{D}, \mathcal{R})$ be the optimal welfare achieved by the set of drivers $\mathcal{D}$ and the set of riders $\mathcal{R}$, i.e. the optimal objective of (5). Moreover, we denote $\tilde{W}(\mathcal{D} \cup \{(a, b, t)\}, \mathcal{R})$ as the optimal objective of (5) if the trip capacity constraint (6) for this specific trip is relaxed by 1, i.e. where there is an additional driver that is only able to complete an $(a, b, t)$ trip.

Similarly, denote $W(\mathcal{D}, \mathcal{R})$ as the optimal objective of (10) and $W(\mathcal{D} \cup \{(a, b, t)\}, \mathcal{R})$ as the optimal objective of (10) with an extra $(a, b, t)$ trip capacity. From the integrality of the LP (10) under (S1) and (S2), we know that $\tilde{W}(\mathcal{D}, \mathcal{R}) = W(\mathcal{D}, \mathcal{R})$, however, we only know $\tilde{W}(\mathcal{D} \cup \{(a, b, t)\}, \mathcal{R}) \leq W(\mathcal{D} \cup \{(a, b, t)\}, \mathcal{R})$ since with the additional capacity of 1 for the $(a, b, t)$ trip, it is not obvious whether the LP relaxation is still integral.

*Part (i):* To prove (i), first observe that with the same argument on subgradients as in the proof of Lemma 4, we can show that under any CE outcome, the price $p_{a,b,t}$ being the subgradient w.r.t. the RHS of capacity constraint (11) in the LP (10), is lower bounded by the welfare gain from relaxing the capacity constraint by 1, i.e. $p_{a,b,t} \geq W(\mathcal{D} \cup \{(a, b, t)\}, \mathcal{R}) - W(\mathcal{D}, \mathcal{R})$. This implies that $p_{a,b,t} \geq \tilde{W}(\mathcal{D} \cup \{(a, b, t)\}, \mathcal{R}) - \tilde{W}(\mathcal{D}, \mathcal{R})$, i.e. any CE price must be at least the welfare contribution of an additional $(a, b, t)$ trip to the original economy.

What is left to prove is that $\tilde{W}(\mathcal{D} \cup \{(a, b, t)\}, \mathcal{R}) - \tilde{W}(\mathcal{D}, \mathcal{R}) \geq p^{vcg}_{a,b,t}$, where $p^{vcg}_{a,b,t} = \tilde{W}(\mathcal{D}, \mathcal{R} \backslash \{j\}) - (\tilde{W}(\mathcal{D}, \mathcal{R}) - v_j)$, i.e. the optimal welfare in the economy without rider $j$ minus the welfare of the rest of the riders in the economy with rider $j$. This holds since:

$$
\begin{aligned}
&\tilde{W}(\mathcal{D} \cup \{(a, b, t)\}, \mathcal{R}) - \tilde{W}(\mathcal{D}, \mathcal{R}) - p^{vcg}_{a,b,t} \\
=&\tilde{W}(\mathcal{D} \cup \{(a, b, t)\}, \mathcal{R}) - \tilde{W}(\mathcal{D}, \mathcal{R}) - (\tilde{W}(\mathcal{D}, \mathcal{R} \backslash \{j\}) - (\tilde{W}(\mathcal{D}, \mathcal{R}) - v_j)) \\
=&\tilde{W}(\mathcal{D} \cup \{(a, b, t)\}, \mathcal{R}) - (\tilde{W}(\mathcal{D}, \mathcal{R} \backslash \{j\}) + v_j) \\
\geq&0.
\end{aligned}
$$

where the last inequality is because $\tilde{W}(\mathcal{D} \cup \{(a, b, t)\}, \mathcal{R})$, the optimal welfare from adding both a trip $(a, b, t)$ and a rider $j$ to the economy $(\mathcal{D}, \mathcal{R} \backslash \{j\})$, is weakly higher than assigning the trip $(a, b, t)$ to rider $j$ and keeping the plan for the rest of the economy unchanged.

This completes the proof of part (i), that any CE price is weakly higher than the VCG payment.

*Part (ii):* Given $\mathcal{D}$ and $\mathcal{R}$, we construct an alternative economy $E' = (\mathcal{D}, \mathcal{R}')$ where $\mathcal{R}'$ and $\mathcal{R}$ coincide, except for the value of rider $j$: instead of having value $v_j$, we change her value to her VCG payment in the original economy, i.e.

$$
v'_j = p^{vcg}_{a,b,t} = \tilde{W}(\mathcal{D}, \mathcal{R} \backslash \{j\}) - (\tilde{W}(\mathcal{D}, \mathcal{R}) - v_j) = \tilde{W}(\mathcal{D}, \mathcal{R} \backslash \{j\}) - \tilde{W}(\mathcal{D} \backslash \{(a, b, t)\}, \mathcal{R} \backslash \{j\}),
$$

Here, $\tilde{W}(\mathcal{D} \backslash \{(a, b, t)\}, \mathcal{R} \backslash \{j\})$ is the highest welfare that can be achieved for the rest of the economy where one of the drivers have to pick up rider $j$.

Now consider the optimal dispatching in the economy $E'$. If rider $j'$ is not picked up, the optimal welfare would be $\tilde{W}(\mathcal{D}, \mathcal{R} \backslash \{j\})$. If rider $j$ is picked up, the optimal welfare would be $v'_j + \tilde{W}(\mathcal{D} \backslash \{(a, b, t)\}, \mathcal{R} \backslash \{j\}) = \tilde{W}(\mathcal{D}, \mathcal{R} \backslash \{j\})$. This implies that in some optimal dispatching in economy $E'$, rider $j'$ in picked up. Let $(x', y')$ be an optimal dispatching in economy $E'$ where rider $j'$ is picked up, and let $p'$ be any competitive equilibrium prices. First, observe that $p'_{a,b,t} \leq v'_j = p^{vcg}_{a,b,t}$, since the outcome forms a CE and rider $j'$ is picked up and must have non-negative utility.

We also claim that the plan with anonymous prices $(x', y', p')$ is also forming a competitive equilibrium in the original economy. Since $(x', y', p')$ as a CE in economy $E'$, we know that for drivers, the dispatched paths under $y'$ gives them the highest total payment given prices $p'$. We also know that trips with excessive supply have zero prices. For any rider $\ell \neq j$, her values in $E'$ and $E$ are the same thus she weakly prefers the outcome $x'_\ell$ to $1 - x'_\ell$. For rider $j$, her value is now $v_j \geq p^{vcg}_{a,b,t} = v'_j \geq p'_{a,b,t}$ thus she weakly prefers $x'_j = 1$ as well. This shows that there exists a CE outcome in $E$ where the price for the $(a, b, t)$ trip is at most $v'_j$, rider $j$'s VCG payment. This completes the proof of part (ii), and also the theorem. □

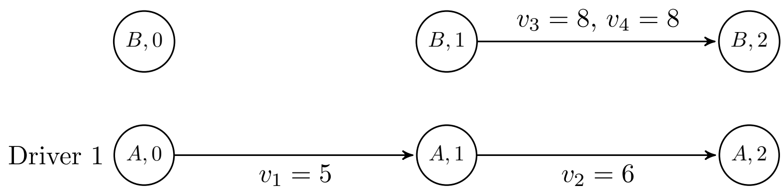



Figure 24: The economy in Example 9, with two locations $A$, $B$, two time periods and four riders.

Finally, we show via the following example that a mechanism that always computes an welfare-optimal dispatching together with rider-side VCG prices (at the beginning of the planning horizon, and also after any driver deviation), is not SPIC for the drivers. This is not implied by Theorem 6 since a mechanism's plan forming a CE is not necessary for a mechanism being SPIC for drivers.

**Example 9.** Consider the economy as shown in Figure 24. The planning horizon is $T = 2$ and there are two locations $\mathcal{L} = \{A, B\}$ with unit distances $\delta(a, b) = 1$ for all $a, b \in \mathcal{L}$. There is one driver entering at time $\underline{\tau}_1 = 0$ at location $\ell_1 = A$ and leaves at time $\bar{\tau}_1 = 2$. There are four riders:

- Rider 1: $o_1 = A$, $d_1 = A$, $\tau_1 = 0$, $v_1 = 5$,
- Rider 2: $o_2 = A$, $d_2 = A$, $\tau_2 = 1$, $v_2 = 6$,
- Rider 3: $o_3 = B$, $d_3 = B$, $\tau_3 = 1$, $v_3 = 8$,
- Rider 4: $o_4 = B$, $d_4 = B$, $\tau_4 = 1$, $v_4 = 8$.

The optimal plan computed at time 0 has driver 1 taking the path $((A, A, 0),\ (A, A, 1))$ and picking up riders 1 and 2. The rider-side VCG prices for riders 1 and 2 would be 2 and 3 respectively, thus the driver's total payment, if she follows the dispatches at all times, would be 5.

Now consider the scenario where the driver relocates to location $B$ at time 0 instead. When time 1 comes, the updated plan would dispatch driver 1 to pick up one of riders 3 or 4, and the updated VCG payment for this trip would be 8. This is, as a result, as useful deviation. □

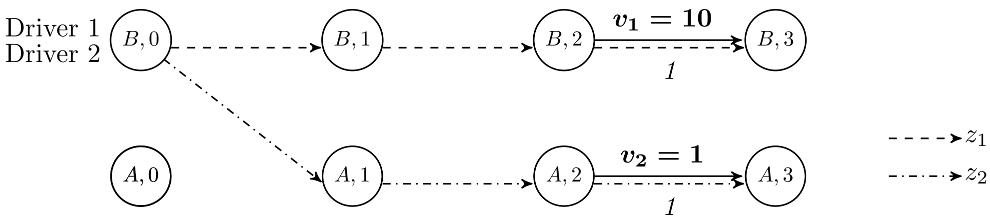
## C.4 Naive Update of Static Plans

The following example shows that a mechanism that always re-computes a driver-optimal plan at all times is not envy-free for the drivers, and may not be incentive compatible, depending on how the mechanism breaks ties among different driver-optimal plans.

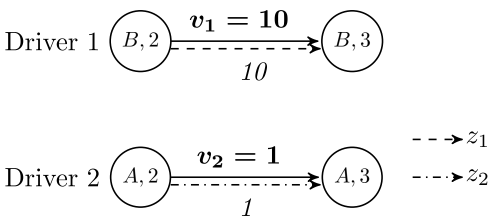
**Example 10.** Consider the economy as illustrated in Figure 25, and a mechanism that repeatedly computes a driver-optimal CE outcome at all times, regardless of whether any deviation happened. Under the CE outcome computed at time 0 as in Figure 25a, both riders 1 and 2 are picked up, and the prices for the trips are both $p_{B,B,2} = p_{A,A,2} = 1$.

Assume that both drivers 1 and 2 follow the mechanism until time 2, the new driver-optimal outcome computed at time 2 would be as illustrated in Figure 25b, where the price for the trip $(B, B, 2)$ becomes 10, the highest market-clearing price at this state. This shows that the "time 0" plan under the mechanism, i.e. the actual outcome where all drivers follow the mechanism's dispatch at all times, is not envy-free for drivers, since the total payment to driver 1 is 10, higher than that of driver 2.

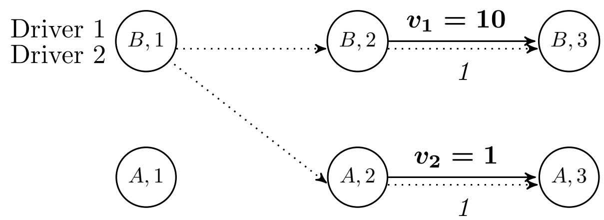
Now consider the scenario where driver 2 stayed in location $B$ at time 0 instead of following the dispatch and relocate to location $A$. Under the optimal CE outcome from time 1 onward as illustrated in Figure 25c, the two drivers take the paths $((B, B, 1),\ (B, B, 2))$ and $((B, A, 1),\ (A, A, 2))$ respectively, and pick up the two drivers. The IC property of the mechanism now depends on how to break ties among driver-optimal CE plans, but as long as the mechanism dispatches driver 2 to take the path $((B, B, 1),\ (B, B, 2))$ with non-zero probability (which would be the case if ties




(a) A driver-optimal CE outcome computed at time 0.




(b) Driver-optimal outcome computed at $t = 2$.




(c) Driver-optimal outcome computed at time $t = 1$ after driver 2's deviation to stay in location $B$.

Figure 25: The economy in Example 10, and driver-optimal CE plans computed at different states.

are broken uniformly at random), this would be a useful deviation for driver 2. Once driver 2 followed the plan and reach $(B, 2)$ and driver 1 arrived at $(A, 2)$, the newly updated price for the trip $(B, B, 2)$ would again become 10. □

# D Relation to the Literature

## D.1 Dynamic VCG

The dynamic VCG mechanisms [6, 12] truthfully implement efficient decision policies, where agent receive private information over time. Under the dynamic VCG mechanisms, the payment to agent $i$ in each period is equal to the flow marginal externality imposed on the other agents by its presence in this period only [12].

The dynamic VCG mechanism can be adapted for the ridesharing problem, where there is no uncertainty in the transition of states (the actions taken by all drivers at time $t$ fully determines the state of the platform at time $t + 1$) and no private information from the drivers' side (the location of the driver is known to the mechanism and we assume homogeneous driver costs and no location preferences). We actually show that a variation of the driver-optimal dynamic mechanism that we discussed in Section 4, where driver payments are "shifted" over time, is equivalent to the dynamic VCG mechanism.

The dynamic VCG mechanism for ridesharing, however, fails to be incentive compatible, since some drivers may be paid a negative payment for certain periods of time, and the drivers would have incentive to decline the dispatch at such times to avoid making the payments. This is because the existence of some driver for only one period of time may exert negative externality on the rest of the economy by inducing seemingly efficient actions that result in suboptimal positioning of the rest of the drivers in the subsequent time periods.

We illustrate this via analyzing the economy introduced in Example 5, as shown in Figure 21.

**Example 5** (Continued)**.** Without driver 1, driver 2 would be dispatched to pick up riders 1 and 2 and achieve a total welfare of 6. With driver 1, one welfare-optimal dispatching plan sends driver 1 to take the path $((B, C, 0),\ (C, C, 1)\ , (C, C, 2))$ and sends driver 2 to take the path $((B, A, 0),\ (A, A, 2))$.

At time 0, driver 1 takes trip $(B, C, 0)$ and driver 2 takes trip $(B, A, 0)$. Driver 1 contributes 0 to welfare at time 0 since she did not pick up any driver. When time 1 comes, if driver 1 appears for only one period of time, the optimal welfare achieved by the rest of the economy would only be 1— driver 1 disappears and driver 2 picked up rider 3. Therefore, the payment to driver 1 at time 0 would be $-5$, since exerted a negative externality of 5 on the rest of the economy by appearing only at time 0. Similarly, we can compute that the payment to driver 1 at times 1 and 2 would be 1 and 5 respectively, giving her a total payment of $-5 + 1 + 5 = 1$.

Now consider the scenario where driver 1 declines the dispatch, refuses to make the payment and stays in location $B$, and assume that driver 2 still followed the mechanism and drove to location $A$. When time 1 comes, driver 1 would again be dispatched to drive to $C$ at time 1 and pick up rider 2 at time 2. We can check that the payment to driver 1 at time 1 would be 0, and the payment to driver 1 at time 2 would be 5— the amount the rest of the economy gains from the existence of driver 2 at that time. This is a useful deviation, thus the dynamic VCG mechanism where drivers are allowed to freely decline trips is not IC. □

## D.2 Trading Networks

The literature on trading networks studies economic models where agents in a network can trade via bilateral contracts [22, 23, 29]. Efficient, competitive equilibrium outcomes exist when agents' valuation functions satisfy the "full substitution" property, and the utilities of agents on either end of the trading network form lattices.

The optimal dispatching problem of ridesharing platforms can be formulated as a trading network problem in the following way:

- For each driver or rider, there is a node in the network.
- For each driver $i \in \mathcal{D}$ and each rider $j \in \mathcal{R}$, there is an edge from $i$ to $j$ if $\tau_j \geq \underline{\tau}_i + \delta(\ell_i, o_j)$, i.e. driver $i$ is able to pick up rider $j$ if she drivers directly to $o_j$ after entering.
- For any two riders $j$ and $j'$ in $\mathcal{D}$, there is an edge from $j$ to $j'$ if (i) $\tau_j + \delta(o_j, d_j) + \delta(d_j, o_{j'}) \leq \tau_{j'}$ assuming $d_j \neq o_{j'}$ or (ii) $\tau_j + \delta(o_j, d_j) \leq \tau_{j'}$ if $d_j = o_{j'}$. Intuitively, riders $j$ can trade to rider $j'$ if a driver is still able to pick up rider $j'$ after dropping off rider $j$.

What is being traded in the network is the right to use the car over the rest of the planning horizon. Each driver is able to trade with at most one rider. A driver's utilities is zero if she does not trade, and her utility is equal to the her payment if she did trade. Each rider values buying the right to use at most one car, and values it at $v_j$. If she did buy the right to use one car, her utility is $v_j$ minus the price she paid to buy the right to use the car, plus the payment she collected from the rider who bought the right to use the car from her. Riders that did not buy a car cannot sell (values such contracts at $-\infty$).

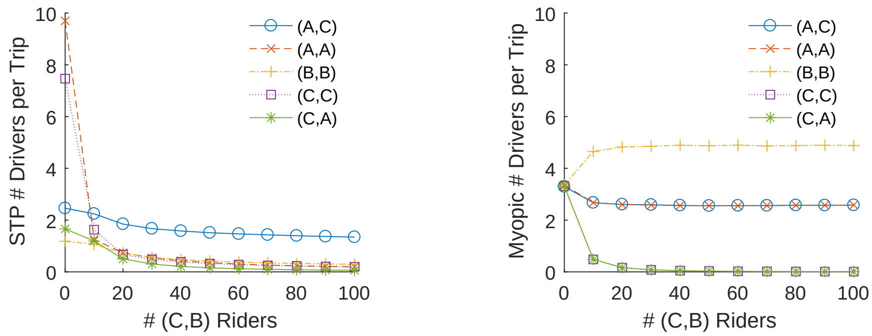
With existing results in the trading network literature, we show the existence of welfare-optimal, competitive equilibrium outcomes, and the lattice structure of drivers' total payments under all CE outcomes. This does not solve our problem, since there isn't language in the trading network literature that describes the temporal evolvement of the network structure and the corresponding incentive issues— as time progresses, the set of reachable riders for each driver decreases, thus the network becomes sparser.

# E Additional Simulation Results

We present in this section the additional simulation results that are omitted from the body of the paper.

## E.1 Morning Rush Hour

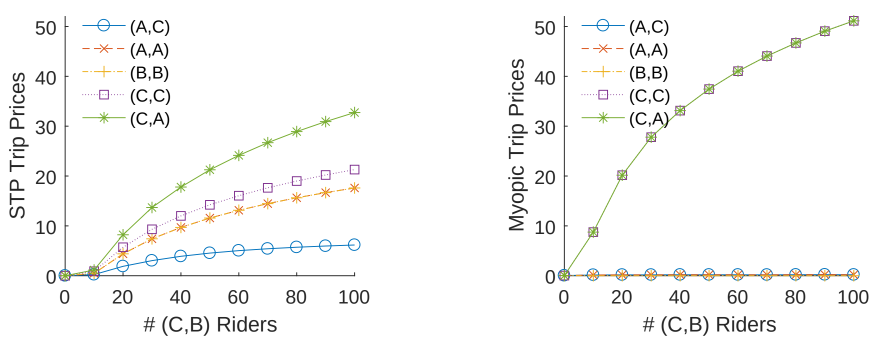
Figures 26 and 27 show the average number of drivers and average prices for each of the five trips that are not analyzed in Section 5.2 for the morning rush hour scenario.




(a) The STP mechanism. (b) The myopic pricing mechanism.

Figure 26: Additional comparison of the number of drivers per trip for the morning rush hour.

The STP mechanism dispatches a reasonably high number of drivers to the $(A, C)$ trip since there is a high demand for drivers at $C$ (see Figure 26). In contrast, though the myopic pricing mechanism is not sending too many drivers from $C$ to $C$ or $A$, many drivers linger around $B$ due to the excessive supply, and the mechanism did not relocate more driver from $A$ to $C$ than from $A$ to $A$, despite the imbalance in demand in these locations. Prices as shown in Figure 27 are also intuitive and easy to interpret.




(a) The STP mechanism. (b) The myopic pricing mechanism.

Figure 27: Additional comparison trip prices for the morning rush hour.