

Prior-free Data Acquisition for Accurate Statistical Estimation

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Abstract

We study a data analyst's problem of acquiring data from self-interested individuals to obtain an accurate estimation of some statistic of a population, subject to an expected budget constraint. Each data holder incurs a cost, which is unknown to the data analyst, to acquire and report his data. The cost can be arbitrarily correlated with the data. The data analyst has an expected budget that she can use to incentivize individuals to provide their data. The goal is to design a joint acquisition-estimation mechanism to optimize the performance of the produced estimator, without any prior information on the underlying distribution of cost and data. We investigate two types of estimations: unbiased point estimation and confidence interval estimation.

Unbiased estimators: We design a truthful, individually rational, online mechanism to acquire data from individuals and output an unbiased estimator of the population mean when the data analyst has no prior information on the cost-data distribution and individuals arrive in a random order. The performance of this mechanism matches that of the optimal mechanism, which knows the true cost distribution, within a constant factor. The performance of an estimator is evaluated by its variance under the worst-case cost-data correlation.

Confidence intervals: We characterize an approximately optimal (within a factor 2) mechanism for obtaining a confidence interval of the population mean when the data analyst knows the true cost distribution at the beginning. This mechanism is efficiently computable. We then design a truthful, individually rational, online algorithm that is only worse than the approximately optimal mechanism by a constant factor. The performance of an estimator is evaluated by its expected length under the worst-case cost-data correlation.

1 Introduction

We study a data analyst’s problem of estimating a population statistic (e.g. mean workout time in November) when data need to be acquired from self-interested data holders and the analyst has an expected budget constraint. Each data holder has a heterogeneous private cost to acquire and report his data (e.g. record duration of each workout in a month and report the total) and needs to be compensated at least by this cost to reveal his data. Individuals cannot fabric their data if they decide to reveal it. Moreover, the values of the data and the private costs can be arbitrarily correlated in the population (e.g. those who work out regularly may use some fitness tracker which automatically records workout durations) and the correlation is unknown to the analyst a priori. A naive way for the analyst to acquire data in this setting is to offer a fixed compensation for each individual’s data. But unless the payment level is higher than everyone’s cost, in which case the analyst may run out of budget quickly and only be able to obtain a small sample, the collected sample will bias toward a low-cost subpopulation. Thus, the problem is how to design a joint pricing-estimation mechanism to get accurate estimations when data holders are strategic.

The problem of purchasing data for unbiased estimation of population mean was first formulated by Roth and Schoenebeck [2012] and then further studied by Chen et al. [2018a]. Both works however assume that the cost distribution is known to the analyst and aim at obtaining an optimal unbiased estimator with minimum worst-case variance for population mean, where the worst-case is over all data-cost distributions consistent with the known cost distribution, subject to an expected budget constraint. The mechanism proposed by Roth and Schoenebeck [2012] achieves optimality approximately, while the mechanism proposed by Chen et al. [2018a] achieves the exact optimality, both under a regularity assumption on the cost distribution. Chen et al. [2018a] also extends the result to linear regression. The high-level idea of both mechanisms is to acquire a data point with reported cost c_i with a positive probability $A(c_i)$ (and some payment that is greater than or equal to c_i), then remove the sampling bias by re-weighting each collected data by $1/A(c_i)$, and finally average the re-weighted data to obtain an unbiased estimation (the Horvitz-Thompson Estimator). The assumption that the cost distribution is known allows the analyst to turn the mechanism design problem into a constrained optimization problem for finding an optimal allocation rule $A(c_i)$.

This work generalizes the prior works in two directions. First, in our setting, the data analyst has no prior information on data holders’ costs. We investigate the design of online mechanisms for unbiased estimation of population mean, with the same goal as in the prior works: minimize the variance of the unbiased estimator subject to a budget constraint. Data holders arrive in a random order. The challenge is that, in order to price well, the analyst needs to learn the cost distribution, but the pricing decisions need to be made for every arriving data holder. Second, we consider the estimation of confidence intervals of population mean, for both the scenario where the analyst knows the cost distribution and the scenario where there is no prior information on costs. Our goal is to minimize the length of the confidence interval given the budget constraint. This necessarily requires us to reason about bias and variance tradeoffs together with data pricing, an aspect that, to the best of our knowledge, has not been explored in the literature.

1.1 Summary of Our Results and Techniques

Our work mainly addresses two questions:

1. If the data analyst does not have any prior information on data holders’ private costs (as well as their private data), is it possible to design an online data acquisition mechanism for unbiased estimation of population mean that is competitive with the optimal mechanism that knows the cost distribution a priori? Optimality refers to minimum variance of the estimator.
2. Can we design an optimal joint acquisition-estimation mechanism for estimating confidence intervals of population mean, when cost distribution is known? Optimality here means minimum length of the confidence interval. When cost distribution is unknown, can we design an online joint acquisition-estimation mechanism for confidence intervals that is competitive with the optimal mechanism that knows the cost distribution a priori?

For the first question, we design an online mechanism that is only worse than the optimal mechanism by a constant factor. The only non-trivial assumption we make in our setting is that the data holders come in random order, so if there are n data holders in total, the cost-data distribution at each round is the discrete uniform distribution over the set of cost-data pairs of these n data holders. Our mechanism

satisfies the budget constraint in expectation, with the guarantee that the data holders will always be willing to participate and truthfully report their costs.

Theorem 1.1 (Informal). *For the problem of purchasing data to get an unbiased estimator of population mean, assuming that the data holders come in random order, our online mechanism (Mechanism 1) satisfies the following properties: (1) it is truthful and individually rational, (2) it satisfies the expected budget constraint, and (3) for any cost distribution, the variance of the produced unbiased estimator approaches that of the benchmark within a constant factor, where the benchmark is the optimal mechanism that knows the true cost distribution a priori.*

In designing our online mechanism (Mechanism 1), we use the optimal mechanism with known costs as our building blocks. At any round i , the reported costs in previous rounds gives us an empirical cost distribution. We then apply the optimal mechanism for this cost distribution for data holder i . Each round's mechanism has a fraction of the total expected budget. Our mechanism allocates more budget for early rounds in a way so that the variance of the final produced estimator is only worse than the benchmark by a constant factor.

For the second question, we extend our mechanism to output a confidence interval (using sample mean and sample variance). The mechanism may introduce some bias to mean estimation in exchange for a lower variance, so that the length of the confidence interval is approximately optimized. We provide the characterization of the approximately optimal confidence interval mechanism when the cost distribution is known. This characterization allows us to efficiently compute the mechanism. We then design an online mechanism that matches the performance of the optimal mechanism that knows cost distribution within a constant factor.

Theorem 1.2 (Informal). *For the problem of purchasing data to obtain a confidence interval, the approximately optimal mechanism that knows cost distribution can be computed in polynomial time.*

This approximately optimal mechanism with known costs is constructed by analyzing the bias and variance trade-off for estimators for the mean. At any given bias level, by producing an estimator that has the lowest variance (for that bias level), we can construct a confidence interval using this biased mean estimation. We hence can design a mechanism to optimize for the length of the confidence interval. Since the optimal mechanism is difficult to compute, we approximate it to gain computational efficiency.

Theorem 1.3 (Informal). *For the problem of purchasing data to obtain a confidence interval, assuming that the data holders come in random order, our online mechanism (Mechanism 2) has the following properties: (1) it is truthful and individually rational, (2) it satisfies the expected budget constraint, and (3) for any cost distribution, the performance of the produced confidence interval approaches that of the benchmark within a constant factor, where the benchmark is the optimal mechanism that knows the true cost distribution a priori.*

Similar to the design of the online mechanism for unbiased mean estimation, we use the approximately optimal mechanism with known cost as our build blocks and design an online mechanism for confidence interval estimation without knowing the costs. Each round is allocated with a fraction of the total expected budget and we take the empirical cost distribution and apply the optimal mechanism for that distribution.

1.2 Other Related Work

There is a growing interest in understanding statistical estimation and learning in environments with strategic agents. The works can be put in a few categories depending on the sources and types of strategic considerations.

In this work, as well as Roth and Schoenebeck [2012] and Chen et al. [2018a], agents do not derive utility or disutility from the estimation outcome, cannot fabricate their data, and have a cost for revealing their data. The mechanism uses payment to incentivize data revelation. Abernethy et al. [2015] is similar on these fronts, but the work considers general supervised learning. They do not seek to achieve a notion of optimality. Instead, they take a learning-theoretic approach and design mechanisms to obtain learning guarantees (risk bounds). Cai et al. [2015] focuses on incentivizing individuals to exert effort to obtain high-quality data for the purpose of linear regression.

Another line of research examines data acquisition using differential privacy [Ghosh and Roth, 2011, Fleischer and Lyu, 2012, Ghosh et al., 2014, Nissim et al., 2014, Cummings et al., 2015]. Agents care about their privacy and hence may be reluctant to reveal their data. The mechanism designer uses

payments to balance the trade-off between privacy and accuracy. In this work, we implicitly assume that data holders do not have a privacy cost and hence they don't worry about potential leaking of their data by reporting their cost. In Section 6, we discuss the complication when data holders care about their privacy and their data and costs are correlated.

A third line of research studies settings where data holders may strategically misreport their data, there is no ground truth to verify the acquired data, and the analyst would like to design payment mechanisms to incentivize truthful data reporting for the purpose of regression or other analyses [Liu and Chen, 2018, 2016, 2017]. Because of the lack of verification, this line of work is closely related to the literature on peer prediction [Miller et al., 2005, Shnayder et al., 2016].

In a fourth line of research, individuals' utilities directly depend on the inference or learning outcome (e.g. they want a regression line to be as close to their own data point as possible) and they can manipulate their reported data to influence the outcome. In these works, there often is no cost for reporting one's data and the data analyst doesn't use monetary payments. These works attempt to design or identify mechanisms (inference or learning processes) that are robust to potential data manipulations [Dekel et al., 2010, Meir et al., 2011, 2012, Perote and Perote-Peña, 2003, 2004, Hardt et al., 2016, Dong et al., 2017, Chen et al., 2018b].

2 Model

Consider a data analyst who conducts a survey to estimate some statistic of a population (or a random sample of a population) of n people. In this work we focus on estimating the mean of some parameter of interest (e.g. alcohol consumption or BMI of an individual), denoted by z , and the confidence interval of the mean. Each individual incurs a cost c_i , unknown to the data analyst, to acquire and report his data z_i . The cost and data pair can be correlated (e.g. those who consume more alcohol may have a higher cost recording their consumption), and follows an unknown distribution \mathcal{D} supported on $(\mathcal{C}, \mathcal{Z})$. We assume that the cost is bounded by \bar{C} , i.e., $\mathcal{C} \subseteq [0, \bar{C}]$. The parameter z is also bounded, and, without loss of generality, we assume z is between 0 and 1, i.e., $\mathcal{Z} \subseteq [0, 1]$. The data analyst has a budget $B = n\bar{B}$ that she can use to purchase data from the data holders.

We study an online setting where data holders arrive one by one to the survey, and no prior information on the distribution \mathcal{D} (including the marginal distribution of the cost) is available before the survey. The analyst can gradually learn the distribution as data holders report their data. We make the following assumptions about the data sequence: (1) each individual only appears once, and (2) the data holders arrive in a *random order*, i.e., each permutation of the n people is equally likely. We use $(\mathbf{c}, \mathbf{z}) = (c_1, z_1), \dots, (c_n, z_n)$ to denote a random sequence of costs and data points, and $\{(c_1, z_1), \dots, (c_n, z_n)\}$ to represent a set of people's cost and data without the consideration of order.

When data holder i arrives, the analyst asks the data holder to report his cost. We use \hat{c}_i to denote the reported cost of data holder i . Based on the reported cost, the analyst may offer a price to acquire the data z_i . Formally, the analyst uses a *survey mechanism*, $M = (A, P)$, which consists of an allocation rule $A : \mathcal{C} \rightarrow [0, 1]$ and a payment rule $P : \mathcal{C} \rightarrow \mathbb{R}$. With probability $A(\hat{c}_i)$, the analyst offers payment $P(\hat{c}_i)$ to purchase data z_i . If the data holder accepts this payment, he gives his data z_i to the analyst. We assume that data holders do not misreport their data z_i . This assumption holds in situations when data can be verified once collected (e.g. medical records). The data holder walks away without revealing his data if $P(\hat{c}_i) < c_i$. With probability $1 - A(\hat{c}_i)$, the analyst does not attempt to acquire the data.

The analyst can adaptively choose a survey mechanism for each arriving data holder. We use $\mathbf{M} = (\mathbf{A}, \mathbf{P}) = (A^1, P^1), \dots, (A^n, P^n)$ to represent a sequence of survey mechanisms. At round i , the analyst chooses an allocation rule A^i and a payment rule P^i based on all observed information before round i , denoted by \mathcal{H}_{i-1} . \mathcal{H}_{i-1} includes the reported costs of the previous $i - 1$ data holders and data points that have been acquired. The survey mechanism (A^i, P^i) applies to the i -th arriving data holder. At the end of round n , the data analyst outputs an estimator $S(\mathbf{M}, (\mathbf{c}, \mathbf{z}))$ based on all observed information \mathcal{H}_n .

We want to design survey mechanisms that have the following incentive and budget properties:

Individual rationality: The utility of each data holder is always non-negative, i.e., $P^i(\hat{c}_i) \geq \hat{c}_i$ for all i and \hat{c}_i .

Truthfulness in expectation: A data holder maximizes his expected utility by reporting his cost truthfully, i.e., $A^i(c_i)(P^i(c_i) - c_i) \geq A^i(\hat{c}_i)(P^i(\hat{c}_i) - c_i)$ for all i and $\hat{c}_i \neq c_i$.

Expected budget feasibility: $\mathbb{E} [\sum_{i=1}^n A^i(c_i) \cdot P^i(c_i)] \leq B = n \cdot \bar{B}$, where the expectation is taken over the random arriving order of the data holders and the internal randomness of the mechanism.

In this work we mainly investigate two types of estimation tasks: (1) get an unbiased estimator of the population mean, with the goal that the variance of the estimator is minimized; (2) find a confidence interval of the population mean, with the goal that the length of the confidence interval is minimized. As an estimator uses data obtained via survey mechanisms \mathbf{M} , it necessarily depends on \mathbf{M} . We now formally define unbiased estimator and confidence interval of population mean in our setting. The randomness of an estimator $S(\mathbf{M}, (\mathbf{c}, \mathbf{z}))$ comes in two parts: (1) the external randomness, which is the random order of (\mathbf{c}, \mathbf{z}) , and (2) the internal randomness of the mechanisms \mathbf{M} . Our definitions require the estimators to be unbiased or a valid confidence interval for any realization of the external randomness.

Definition 2.1 (Unbiased estimator of population mean). *An estimator $S(\mathbf{M}, (\mathbf{c}, \mathbf{z}))$ is an unbiased estimator of the population mean $\mathbb{E}[z] = \frac{1}{n} \sum_{i=1}^n z_i$ if for any fixed sequence $(\tilde{\mathbf{c}}, \tilde{\mathbf{z}})$,*

$$\mathbb{E}[S(\mathbf{M}, (\tilde{\mathbf{c}}, \tilde{\mathbf{z}}))] = \mathbb{E}[z],$$

where the expectation in $\mathbb{E}[S(\mathbf{M}, (\tilde{\mathbf{c}}, \tilde{\mathbf{z}}))]$ is taken over the internal randomness of the mechanisms \mathbf{M} .

Definition 2.2 (Confidence interval of population mean). *An estimator $S(\mathbf{M}, (\mathbf{c}, \mathbf{z}))$ is a confidence interval for the population mean $\mathbb{E}[z] = \frac{1}{n} \sum_{i=1}^n z_i$ with confidence level γ if it is an interval and for any fixed sequence $(\tilde{\mathbf{c}}, \tilde{\mathbf{z}})$,*

$$\Pr(\mathbb{E}[z] \in S(\mathbf{M}, (\tilde{\mathbf{c}}, \tilde{\mathbf{z}}))) \geq \gamma,$$

where the randomness is due to the internal randomness of the mechanisms \mathbf{M} .

Our goal is to design joint survey and estimation mechanisms, $(\mathbf{M}, S(\mathbf{M}, (\mathbf{c}, \mathbf{z})))$, such that the estimator $S(\mathbf{M}, (\mathbf{c}, \mathbf{z}))$ has good statistical performance on the population. For unbiased estimators, we prefer estimators with smaller variance. For confidence intervals, we prefer ones with smaller length. However, the performance of a mechanism on a population depends on the correlation between the population's cost and data, i.e. the distribution \mathcal{D} .¹ We hence take a worst-case analysis approach: measure the performance of a mechanism under worst-case cost-data correlation.

Definition 2.3 (Worst-case variance). *Given that the set of data holders' costs is $C = \{c_1, \dots, c_n\}$, the worst-case variance of a point estimator $S(\mathbf{M}, (\mathbf{c}, \mathbf{z}))$ is defined as*

$$\text{Var}^*(S) = \max_{\mathcal{D} \text{ consistent with } C} \text{Var}_{\mathcal{D}}(S(\mathbf{M}, (\mathbf{c}, \mathbf{z})))$$

where the maximum is taken over all distributions \mathcal{D} consistent with the set of costs C . The randomness is due to the random order of $(c_1, z_1), \dots, (c_n, z_n)$ and the internal randomness of the mechanism \mathbf{M} .

Definition 2.4 (Worst-case expected length). *Given that the set of data holders' costs is $C = \{c_1, \dots, c_n\}$, the worst-case expected length of a confidence interval $S(\mathbf{M}, (\mathbf{c}, \mathbf{z}))$ is defined as*

$$L^*(S) = \max_{\mathcal{D} \text{ consistent with } C} \mathbb{E}(|S(\mathbf{M}, (\mathbf{c}, \mathbf{z}))|)$$

where $|S(\mathbf{M}, (\mathbf{c}, \mathbf{z}))|$ represents the length of the confidence interval. The maximum is taken over all distributions \mathcal{D} consistent with the set of costs C . The randomness is due to the random order of $(c_1, z_1), \dots, (c_n, z_n)$ and the internal randomness of the mechanisms \mathbf{M} .

Roth and Schoenebeck [2012] and Chen et al. [2018a] have also considered the design of joint survey and estimation mechanism for statistical estimation. The main differences between their model and our model are: (1) they assume the marginal cost distribution is known to the data analyst, while our data analyst doesn't have such information, (2) they have the same survey mechanism for all individuals, while we consider an online setting where the analyst can adaptively change the survey mechanism, and (3) they only consider the estimation of mean, while we also investigate the estimation of confidence intervals.

¹ For example, consider a mechanism that purchases each agent's data z_i with a constant probability $p = B / \sum_{i=1}^n c_i$ and payment c_i , then outputs $1/(pn)$ times the sum of all purchased data as an unbiased estimation of population mean. When z is always equal to 0, the variance will be zero; when z is always equal to 1, the variance will be $(1/p - 1)/n$.

3 Preliminaries

In this section, we first show that we can easily extend known results on one-shot truthful mechanisms to achieve truthfulness and individual rationality for a sequence of survey mechanisms \mathbf{M} . Then, we introduce the formulation proposed by [Chen et al. \[2018a\]](#) for obtaining the optimal unbiased estimator of population mean *when the cost distribution is known to the analyst*. Later in Section 4 we will use this known cost case as our benchmark for evaluating the performance of our optimal unbiased estimator when the cost distribution is unknown.

3.1 Truthful and Individually Rational Survey Mechanisms

Since each data holder appears only once in our setting, requiring a sequence of survey mechanisms to be truthful and individually rational is equivalent to requiring that each (A^i, P^i) is truthful and individually rational, which can be achieved by a straight-forward extension of known results on truthful mechanisms.

The well-known Myerson's lemma states that monotonicity is the necessary and sufficient condition for an allocation rule to be truthful with some payment rule.

Lemma 3.1 ([Myerson and Satterthwaite \[1983\]](#)). *An allocation rule $A(c)$ is the allocation rule of some truthful survey mechanism $(A(c), P(c))$ if and only if $A(c)$ is monotone non-increasing in c .*

The following lemma from [Chen et al. \[2018a\]](#) then shows that given a fixed monotone non-increasing allocation rule $A(c)$ defined on a discrete cost set $\{c_1, \dots, c_m\}$, there exists an optimal payment rule $P(c)$ that guarantees truthfulness and individual rationality.

Lemma 3.2 ([Chen et al. \[2018a\]](#), Claim 2 in Section B.1.2). *Let $A(c)$ be a monotone non-increasing allocation rule defined for set $\{c_1, \dots, c_m\}$ with $c_1 \leq \dots \leq c_m$. Define payment rule $P(c_i) = c_i + \frac{1}{A(c_i)} \sum_{j=i+1}^m A(c_j)(c_j - c_{j-1})$. Then $(A(c), P(c))$ is truthful and individually rational for all $c \in \{c_1, \dots, c_m\}$, and any payment rule $P'(c)$ that guarantees the truthfulness and individual rationality of $(A(c), P'(c))$ must have $P'(c) \geq P(c)$ for all $c \in \{c_1, \dots, c_m\}$.*

Furthermore for any cost distribution supported on $\{c_1, \dots, c_m\}$, the expected total payment of $(A(c), P(c))$, with the optimal payment rule $P(c)$ defined in Lemma 3.2, can be equivalently represented in a simpler form in terms of virtual costs.

Definition 3.1 (Virtual costs). *Let $f(c)$ and $F(c)$ be the PDF and the CDF of a cost distribution \mathcal{F} supported on $\{c_1, \dots, c_m\}$ with $c_1 \leq \dots \leq c_m$. Let $c_0 = 0$. The virtual cost function $\psi(c)$ of this cost distribution is defined as*

$$\psi(c_i) = c_i + \frac{c_i - c_{i-1}}{f(c_i)} F(c_{i-1})$$

for all $1 \leq i \leq m$.

Lemma 3.3 ([Chen et al. \[2018a\]](#), Lemma 10 in Section B.1.2). *Let $A(c)$ be a monotone non-increasing allocation rule defined on set $\{c_1, \dots, c_m\}$ with $c_1 \leq \dots \leq c_m$. Let $P(c)$ be the optimal truthful and individually rational payment rule defined in Lemma 3.2. When cost follows a distribution \mathcal{F} supported on $\{c_1, \dots, c_m\}$, the expected total payment $\mathbb{E}_{c \sim \mathcal{F}}[A(c)P(c)]$ is equal to the expected virtual cost $\mathbb{E}_{c \sim \mathcal{F}}[A(c)\psi(c)]$ where $\psi(c)$ is the virtual cost function of \mathcal{F} .*

The above lemmas assume that costs are from a finite discrete set. Our benchmark mechanism where the analyst already knows all the costs satisfies this assumption. We'll use the above result to establish the performance of our benchmark mechanism. However, our mechanisms developed in this paper for the unknown cost case do not have any restriction on the set of possible costs. The allocation rules and the payment rules of our mechanisms are first computed on a discrete set using the above result, and then extended to all other values of cost. We show below that such extension preserves truthfulness and individual rationality.

Definition 3.2 (Extended allocation rule and payment rule). *Given a survey mechanism (A^d, P^d) that is defined on a discrete cost set $\{c_1, \dots, c_m\}$ with $c_1 \leq \dots \leq c_m$. The extended allocation rule and payment rule A, P are defined as follows*

$$A(c) = A^d(\lceil c \rceil), \quad P(c) = P^d(\lceil c \rceil), \quad \text{for all } c \in [0, c_m],$$

where $\lceil c \rceil$ is the minimum cost in $\{c_1, \dots, c_m\}$ that is greater than or equal to c .

Lemma 3.4. Let $A^d(c)$ be a monotone non-increasing allocation rule defined on set $\{c_1, \dots, c_m\}$ with $c_1 \leq \dots \leq c_m$. Let $P^d(c)$ be the optimal payment rule defined in Lemma 3.2. Then the extended allocation rule and payment rule of (A^d, P^d) is still truthful and individually rational.

The lemma is proved in B.1.

3.2 Formulating the Optimal Unbiased Estimator with Known Costs

In this section we restate some results from Chen et al. [2018a] on the optimal survey mechanism for an unbiased estimator of population mean *when the cost distribution is known*. The value of statistic z is assumed to be bounded and without loss of generality $0 \leq z \leq 1$.

Horvitz-Thompson Estimator: When we use truthful survey mechanisms $\mathbf{M} = (A^1, P^1), \dots, (A^n, P^n)$ to purchase the data points, the data of agent i will be collected with probability $A^i(c_i)$. Define

$$\hat{x}_i = \begin{cases} z_i, & \text{with probability } A^i(c_i) \\ 0, & \text{otherwise} \end{cases}$$

to be the observed data point which is set to zero if no purchase is made. Define $y_i = \frac{\hat{x}_i}{A^i(c_i)}$. To get unbiased estimation, we use Horvitz-Thompson Estimator, which is the unique unbiased linear estimator in our setting [Roth and Schoenebeck, 2012],

$$S(\mathbf{M}, (\mathbf{c}, \mathbf{z})) = \frac{1}{n} \sum_{i=1}^n y_i.$$

Notice that an unbiased estimator always buys the data points with probability greater than 0, i.e., $A^i(c_i) > 0$ for all i and c_i . If $A^i(c_i) = 0$, the mechanism will never know the value of z_i and thus cannot be unbiased.

When the cost distribution is known to the analyst, the optimal mechanism that uses the same survey mechanism for all data holders has been derived by Chen et al. [2018a]. They reduce the mechanism design problem to a min-max optimization problem. The optimal allocation rule A^* that minimizes the worst-case variance of the Horvitz-Thompson Estimator can be formulated as follows:

$$\begin{aligned} \arg \min_A \max_{\mathbf{z} \in [0,1]^n} & \frac{1}{n^2} \left(\sum_{i=1}^n \frac{z_i^2}{A(c_i)} - \sum_{i=1}^n z_i^2 \right) \\ \text{s.t.} & \sum_{i=1}^n A(c_i) \psi(c_i) \leq B \\ & A(c_i) \geq A(c_{i+1}), \quad \forall 1 \leq i < n \\ & 0 \leq A(c) \leq 1, \quad \forall c \end{aligned} \tag{1}$$

Here the objective function is changed from the original formulation in Chen et al. [2018a] so that it is equal to the worst-case variance of the Horvitz-Thompson Estimator in our setting. According to the law of total variance,

$$\text{Var}(S(\mathbf{M}, (\mathbf{c}, \mathbf{z}))) = \mathbb{E}[\text{Var}(S|(\mathbf{c}, \mathbf{z}))] + \text{Var}(\mathbb{E}[S|(\mathbf{c}, \mathbf{z})])$$

Since the estimator is always unbiased for any order (\mathbf{c}, \mathbf{z}) , the second term is zero. Furthermore, when conditioning on a sequence, $y_i = \frac{\hat{x}_i}{A(c_i)}$ become independent when a fixed allocation rule A is used. Therefore the variance of the Horvitz-Thompson Estimator is equal to

$$\begin{aligned} \text{Var}(S(\mathbf{M}, (\mathbf{c}, \mathbf{z}))) &= \mathbb{E}_{(\mathbf{c}, \mathbf{z})}[\text{Var}(S|(\mathbf{c}, \mathbf{z}))] = \mathbb{E}_{(\mathbf{c}, \mathbf{z})} \left[\frac{1}{n^2} \sum_{i=1}^n \text{Var}(y_i|(\mathbf{c}, \mathbf{z})) \right] \\ &= \frac{1}{n^2} \cdot \mathbb{E}_{(\mathbf{c}, \mathbf{z})} \left[\sum_{i=1}^n \mathbb{E}[y_i^2|(\mathbf{c}, \mathbf{z})] - \mathbb{E}[y_i|(\mathbf{c}, \mathbf{z})]^2 \right]. \end{aligned}$$

For any arriving sequence (\mathbf{c}, \mathbf{z}) , $\sum_{i=1}^n \mathbb{E}[y_i^2 | (\mathbf{c}, \mathbf{z})] - \mathbb{E}[y_i | (\mathbf{c}, \mathbf{z})]^2$ stays the same, which is equal to $\sum_{i=1}^n \frac{z_i^2}{A(c_i)} - \sum_{i=1}^n z_i^2$. Therefore by maximizing over \mathbf{z} , we get the worst-case variance of the Horvitz-Thompson estimator

$$\max_{\mathbf{z} \in [0,1]^n} \frac{1}{n^2} \left(\sum_{i=1}^n \frac{z_i^2}{A(c_i)} - \sum_{i=1}^n z_i^2 \right).$$

The last constraint $0 \leq A(c) \leq 1$ makes sure that A is an allocation rule of a survey mechanism. The second constraint is the sufficient and necessary condition for A to be the allocation rule of a truthful mechanism. The first constraint guarantees expected budget feasibility according to Lemma 3.3, where $\psi(c)$ is the virtual cost function.

4 Optimal Unbiased Estimator

We first introduce the benchmark to which we compare our online algorithm.

Definition 4.1. Let $c_{(1)} \leq \dots \leq c_{(n)}$ be the n data holders' costs ordered from smallest to largest. Suppose there is one more data holder with cost \bar{C} . We define the benchmark (A^*, P^*) to be the mechanism that purchases data from these $n+1$ data holders, and minimizes the worst case variance when it knows the set of costs $\{c_{(1)}, c_{(2)}, \dots, c_{(n)}, \bar{C}\}$ at the beginning.

$$\begin{aligned} \text{Var}^*(A^*) &= \frac{1}{(n+1)^2} \min_A \max_{\mathbf{z} \in [0,1]^n} \sum_{i=1}^{n+1} \frac{z_i^2}{A(c_i)} - \sum_{i=1}^{n+1} z_i^2 \\ &\text{s.t.} \quad \sum_{i=1}^{n+1} A(c_i) \psi(c_i) \leq B \\ &\quad A(c_i) \geq A(c_{i+1}), \quad \forall 1 \leq i \leq n \\ &\quad 0 \leq A(c) \leq 1, \quad \forall c \end{aligned} \tag{2}$$

where $c_{n+1} = \bar{C}$.

This additional cost \bar{C} can be interpreted as the loss of unknown upper bound of the possible cost. When the cost distribution is known, the mechanism knows the exact maximum cost $c_{(n)}$, and thus the optimal mechanism will never have a positive probability to buy a data point with cost higher than $c_{(n)}$. But when $c_{(n)}$ is unknown, the mechanism always has to buy any data point (with cost under \bar{C}) with a positive probability.

When the cost distribution is unknown at the beginning, the idea of our mechanism is very simple: at time i , use the optimal allocation rule A^i as if the cost distribution is the uniform distribution on all the observed costs c_1, \dots, c_{i-1} plus $\{\bar{C}\}$, and the budget per agent is proportional to $\frac{1}{\sqrt{i}}$. So the average budget is a decreasing function of i , which means we use more budget at the beginning of the mechanism when the estimation of the cost distribution is not accurate. Instead of solving the exact optimal A^i , we solve an approximation of the optimization problem defined in (2) by replacing the objective function $\text{Var}(S) = \mathbb{E}[S^2] - \mathbb{E}[S]^2$ by $\mathbb{E}[S^2]$ for simple analysis. The payment rule P^i can be computed as in Lemma 3.2. Our mechanism is described as Mechanism 1.

The optimization problem (3) has a convex objective function and thus can be solved efficiently. Below we give the exact characterization of the optimal solution, which has a very simple form and will further be used to derive our optimal confidence interval mechanism: the optimal allocation rule $A^i(c)$ is inversely proportional to the square root of the regularized virtual cost of c , which is defined as follows,

Definition 4.2 (Regularized virtual costs). For a discrete uniform distribution supported on $\{c_1, \dots, c_m\}$ with $c_1 \leq \dots \leq c_m$ and its virtual costs function $\psi(c_1), \dots, \psi(c_m)$. For every $i \leq k$, let $\text{Avg}(i, k)$ be the average of $\psi(c_i), \dots, \psi(c_k)$. We define regularized virtual cost $\phi(c_i)$ as follows

$$\begin{aligned} \phi(c_i) &= \max\{\psi'(c_1), \dots, \psi'(c_i)\}, \\ \psi'(c_i) &= \min_{k:k \geq i} \text{Avg}(i, k). \end{aligned}$$

Mechanism 1: Mechanism for unbiased estimator

Input total number of agents n and expected budget $B = n\bar{B}$.

for $i = 1, \dots, n$ **do**

1) Define set $T_i = \{c_1, \dots, c_{i-1}, \bar{C}\}$ for $i \geq 2$ and $T_i = \{\bar{C}\}$ for $i = 1$. We use $c_{(1)}, \dots, c_{(i)}$ to denote the elements in T_i ordered from smallest to largest.

2) Let A^i be the optimal allocation when the set of costs is T_i , the budget per agent is $\frac{B}{4\sqrt{n/i}}$, i.e.,

$$\begin{aligned}
 A^i &= \arg \min_A \max_{\mathbf{z} \in [0,1]^i} \sum_{k=1}^i \frac{z_k^2}{A(c_{(k)})} & (3) \\
 \text{s.t. } & \sum_{k=1}^i A(c_{(k)}) \psi^i(c_{(k)}) \leq \frac{B}{4\sqrt{n/i}} \\
 & A(c_{(k)}) \geq A(c_{(k+1)}), \forall k \\
 & 0 \leq A(c_{(k)}) \leq 1, \quad \forall k
 \end{aligned}$$

where $\psi^i(c)$ is the virtual cost function (Definition 3.1) when the cost distribution is the uniform distribution over T_i . The payment P^i is computed as in Lemma 3.2.

3) Ask agent i to report his cost c_i and purchase agent i 's data using the extended allocation rule and payment rule of (A^i, P^i) . Let the collected data be \hat{x}_i .

end for

Output estimator $S = \frac{1}{n} \sum_{i=1}^n y_i = \frac{1}{n} \sum_{i=1}^n \frac{\hat{x}_i}{A^i(c_i)}$.

Theorem 4.1. *The optimal solution of (3) is*

$$A^i(c_{(k)}) = \min \left\{ 1, \frac{\lambda}{\sqrt{\phi^i(c_{(k)})}} \right\}, \quad \text{for all } 1 \leq k \leq i,$$

where $\phi^i(c)$ is the regularized virtual cost function when the cost distribution is the uniform distribution over T_i , and λ is chosen such that the budget constraint is satisfied with equality

$$\sum_{j=1}^i A^i(c_{(j)}) \psi^i(c_{(j)}) = \frac{B}{4\sqrt{n/i}}.$$

The value of λ can be computed using binary search within time $O(\log |T_i|)$.

Proof of the theorem can be found in C.1.

We show that this simple mechanism satisfies all three constraints and its worst-case variance is roughly within a constant times the benchmark.

Theorem 4.2. *Mechanism 1 satisfies (1) truthfulness and individually rationality, (2) the expected total spending is no more than $B = n\bar{B}$, and (3) for any cost distribution $\{c_1, \dots, c_n\}$ the worst-case variance of the final estimator S is no more than*

$$16 \cdot \left(\left(1 + \frac{1}{n} \right)^2 \cdot \text{Var}^*(A^*) + \frac{1}{n} + \frac{1}{n\sqrt{n}} \cdot \frac{1}{A^*(\bar{C})} \right),$$

where A^* is the benchmark defined in Definition 4.1.

Discussion: We have the factor $(1 + \frac{1}{n})^2$ in the first term of our upper bound because the benchmark mechanism has one more data point. It is no more than 4 when $n \geq 1$ and goes to 1 when n gets large. The second additive term $\frac{1}{n}$ is due to our estimation of $\text{Var}(S)$ by $\mathbb{E}[S^2]$. We know that $\text{Var}^*(A^*)$ is roughly $\frac{1}{n+1} \cdot \mathbb{E}[\frac{1}{A^*(c)}]$. So when the problem is non-trivial, we should have the average $\frac{1}{A^*(c)}$ much larger than 1, and $\frac{1}{n}$ will be small compared to $\text{Var}^*(A^*)$. The last additive term $\frac{1}{n\sqrt{n}} \cdot \frac{1}{A^*(\bar{C})}$ is dominated by $\frac{1}{n}$. It is only comparable to $\text{Var}^*(A^*)$ when $\sqrt{n} \leq \frac{1}{A^*(\bar{C}) \cdot \mathbb{E}[\frac{1}{A^*(c)}]}$.

Proof sketch: The key idea of proving the performance and budget feasibility is to compare both of our online algorithm and the benchmark with an intermediate mechanism (A', P') at each step i . This intermediate mechanism (A', P') is basically the same as (A^i, P^i) , but is “one-step-ahead”. Loosely speaking, (A', P') is the optimal mechanism when the same amount of budget is assigned at round i , but knows an additional piece of information, the value of c_i , beforehand. We first show that the difference of (A', P') and (A^i, P^i) can be bounded by a constant factor, and then prove that the worst-case variance of (A', P') is no worse than a constant times the benchmark. The first bound is mainly due to the similarity of (A', P') and (A^i, P^i) . For the second bound, informally speaking, we show that the variance occurred by A' at data point i is no more than $D \cdot \frac{\sqrt{n}}{\sqrt{i}}$ times that of A^* (mainly because of the budget allocation method and the random arriving order), where D is a constant. Then, because $\frac{1}{n} \sum_{i=1}^n \frac{\sqrt{n}}{\sqrt{i}} \leq 2$, we bound the difference within a constant factor.

The complete proof can be found in Appendix C.3.

5 Optimal Confidence Interval

In this section, we design purchasing mechanisms to get the best confidence interval of the statistic. We consider the class of confidence intervals that are defined around the sample mean, and the length of which is decided by a bias term and the sample variance. In this case, the optimal mechanism needs to find the optimal trade-off of the bias and the variance in order to minimize the length of the interval. We first present the optimal mechanism when the cost distribution is known to the analyst at the beginning of the survey, and then introduce an online mechanism which is only worse by a constant factor.

5.1 Confidence Interval and Bias-variance Tradeoff

In this paper, we use the most classic approach to construct confidence interval based on sample mean and sample variance. We first show how to convert our unbiased estimator into a confidence interval using sample mean and sample variance.

Construct confidence interval using unbiased estimator: Consider an unbiased estimator $S(\mathbf{M}, (\mathbf{c}, \mathbf{z}))$ that uses survey mechanism $\mathbf{M} = (\mathbf{A}, \mathbf{P})$, and we want to construct a confidence interval for $\mathbb{E}[z] = \frac{1}{n} \sum_{i=1}^n z_i$. Again we use

$$\hat{x}_i = \begin{cases} z_i, & \text{with probability } A^i(c_i) \\ 0, & \text{otherwise} \end{cases}$$

to denote the observed data point and define $y_i = \frac{\hat{x}_i}{A^i(c_i)}$. Notice that the random variables y_1, \dots, y_n are not independent since the allocation rule A^i can depend on c_1, \dots, c_{i-1} . But if we consider a fixed realization $(\tilde{\mathbf{c}}, \tilde{\mathbf{z}})$, the mechanisms $(A^1, P^1), \dots, (A^n, P^n)$ will also be fixed. Then y_1, \dots, y_n become independent, because when the probability of purchasing each data point $A^1(c_1), \dots, A^n(c_n)$ is fixed, whether to purchase each data point or not is independently decided. Therefore given a confidence level γ , we can construct a confidence interval of the expected mean $\mathbb{E}[\sum y_i/n | (\tilde{\mathbf{c}}, \tilde{\mathbf{z}})]$ using the sample mean $\sum_{i=1}^n y_i/n$ and sample variance $\hat{\sigma}^2 = \sum_{i=1}^n (y_i - \sum_{i=1}^n y_i/n)^2 / (n-1)$ according to Bernstein’s inequality (see more details in A.1):

$$\left[\sum_{i=1}^n y_i/n - \frac{\alpha_\gamma}{\sqrt{n}} \cdot \hat{\sigma}, \quad \sum_{i=1}^n y_i/n + \frac{\alpha_\gamma}{\sqrt{n}} \cdot \hat{\sigma} \right]$$

where α_γ is a constant that is chosen to achieve confidence level γ . When the estimator is unbiased, $\mathbb{E}[\sum y_i/n | (\tilde{\mathbf{c}}, \tilde{\mathbf{z}})] = \mathbb{E}[z]$ for all $(\tilde{\mathbf{c}}, \tilde{\mathbf{z}})$, this interval is just a confidence interval of $\mathbb{E}[z]$ with confidence level γ .

But this confidence interval may not be optimal. We can allow the mechanism to simply ignore some data points, i.e., to have $A^i(c_i) = 0$ for some i and simply set $y_i = 0$. This can probably reduce the variance of the estimator. But at the same time the estimator becomes biased. We need to increase the length of the confidence interval to include this bias.

Survey mechanisms that allow bias: We add a new component $\mathbf{U} = (U^1, \dots, U^n)$ into our allocation rule to allow biased estimation, where each U^i is a function of reported cost c_i . A mechanism that allows bias consists of $(\mathbf{A}, \mathbf{U}, \mathbf{P})$. When a data point with cost c_i comes at time i , the mechanism first flips a coin \hat{U}_i to decide whether to ignore this data point or not, and the probability of \hat{U}_i being 1

(which means ignoring the data) is equal to $U^i(c_i)$. If the data is ignored, a bias term will be added into the final estimation to compensate the error. If $\widehat{U}_i = 0$, then the mechanism purchases the data with probability $A^i(c_i) > 0$ and pays $P^i(c_i)$ if the data is purchased. Then the observed data \widehat{x}_i follows

$$\widehat{x}_i = \begin{cases} z_i, & \text{with probability } (1 - U^i(c_i))A^i(c_i) \\ 0, & \text{with probability } (1 - U^i(c_i))(1 - A^i(c_i)) \\ \text{ignored,} & \text{with probability } U^i(c_i). \end{cases}$$

We re-define y_i as

$$y_i = \begin{cases} \frac{\widehat{x}_i}{A^i(c_i)}, & \text{if } \widehat{U}_i = 0 \\ 0, & \text{if } \widehat{U}_i = 1. \end{cases}$$

Then for a fixed arriving sequence $(\tilde{\mathbf{c}}, \tilde{\mathbf{z}})$, the bias of estimator $\sum y_i/n$ is equal to

$$err = \mathbb{E}[z] - \frac{1}{n} \sum_{i=1}^n \mathbb{E}[y_i | (\tilde{\mathbf{c}}, \tilde{\mathbf{z}})] = \frac{1}{n} \sum_{i=1}^n z_i - \frac{1}{n} \sum_{i=1}^n (1 - \widehat{U}_i) z_i = \frac{1}{n} \sum_{i=1}^n \tilde{z}_i \cdot \widehat{U}_i.$$

Notice that this bias is not observable because the mechanism does not know the \tilde{z}_i that is not purchased. But since \tilde{z}_i is between 0 and 1 and we use worst-case analysis in this work, we can just assume \tilde{z}_i equals its worst-case value 1. (This can be seen more clearly in our formulation of the optimization problem in the next section.) Then the confidence interval of $\mathbb{E}[z]$ with confidence level γ can be constructed as follows

$$\left[\sum_{i=1}^n y_i/n - \frac{\alpha_\gamma}{\sqrt{n}} \cdot \widehat{\sigma}, \quad \sum_{i=1}^n y_i/n + \frac{1}{n} \sum_{i=1}^n \widehat{U}_i + \frac{\alpha_\gamma}{\sqrt{n}} \cdot \widehat{\sigma} \right]$$

where \widehat{U}_i is the indicator of whether the i -th data point is ignored and $\widehat{\sigma}^2$ is the sample variance of y_1, \dots, y_n .

Although a new component is added to the mechanism, the results in Section 3 can nevertheless be applied to these mechanisms by seeing $(1 - U(c))A(c)$ as the allocation rule.

For convenience, in the rest of the paper, we write $U_c^i = U^i(c)$ for short, and in cases when the costs are indexed as c_1, \dots, c_n or $c_{(1)}, \dots, c_{(n)}$, we use U_j^i to represent $U^i(c_j)$ or $U^i(c_{(j)})$.

5.2 Benchmark: Known Cost Distribution

Our benchmark for the online mechanism is again the optimal mechanism that knows the cost distribution at the beginning and uses a single optimal mechanism (A^*, U^*, P^*) throughout the survey. We still add an additional cost \overline{C} into the underlying cost set of the benchmark mechanism, in order to make the comparison possible without knowing the exact maximum cost. This optimal mechanism should find the optimal trade-off of the bias and variance so as to minimize the worst-case expected length of the confidence interval defined in the previous section. We formulate a min-max optimization problem that approximately solves the optimal allocation rule.

The expected length of the interval we construct is $2 \cdot \frac{\alpha_\gamma}{\sqrt{n}} \cdot \mathbb{E}[\widehat{\sigma}] + \mathbb{E}[err]$. Since the expectation of sample standard deviation $\mathbb{E}[\widehat{\sigma}]$ is difficult to compute, we estimate $\mathbb{E}[\widehat{\sigma}]$ with $\sqrt{\mathbb{E} \sum_{i=1}^n y_i^2/n}$. When $0 \leq \mathbb{E}[y_i] \leq 1$, the difference between $\mathbb{E}[\widehat{\sigma}]$ and $\sqrt{\mathbb{E} \sum_{i=1}^n y_i^2/n}$ is no more than $1 + O(1/n)$ (see A.2 for more details).

The approximate expected length of the confidence interval can thus be written into a function of A and \mathbf{z}

$$2 \cdot \frac{\alpha_\gamma}{\sqrt{n}} \cdot \sqrt{\mathbb{E} \sum_{i=1}^n y_i^2/n + \mathbb{E}[err]} = 2 \cdot \frac{\alpha_\gamma}{\sqrt{n}} \cdot \sqrt{\frac{1}{n} \sum_{i=1}^n \frac{(1 - U_i) z_i^2}{A(c_i)} + \frac{1}{n} \sum_{i=1}^n z_i \cdot U_i}.$$

Then we only need to take maximum over all possible \mathbf{z} to get the worst-case expected length. Suppose the underlying cost set of the benchmark mechanism is $\{c_{(1)}, \dots, c_{(n)}, \overline{C}\}$ with $c_{(1)} \leq \dots \leq c_{(n)}$, then

the approximately optimal allocation rule can be formulated as follows:

$$\begin{aligned}
A^*, U^* = \arg \min_{A, U} \max_{\mathbf{z} \in [0, 1]^n} & 2 \cdot \frac{\alpha_\gamma}{\sqrt{n+1}} \cdot \sqrt{\frac{1}{n+1} \cdot \sum_{i=1}^{n+1} \frac{(1-U_i)z_i^2}{A(c_{(i)})} + \frac{\sum_{i=1}^{n+1} z_i \cdot U_i}{n+1}} \\
\text{s.t.} & \sum_{i=1}^{n+1} (1-U_i) \cdot A(c_{(i)}) \psi(c_{(i)}) \leq B \\
& (1-U_c)A(c) \text{ is monotone non-increasing in } c \\
& 0 \leq A(c) \leq 1, \quad 0 \leq U_c \leq 1, \quad \forall c
\end{aligned} \tag{4}$$

Lemma 5.1. *Let L^* be the value of the objective function (4) when A^* and U^* is used. (L^* is an approximation of the worst-case expected length of the confidence interval produced by (A^*, U^*) .) Then the difference between L^* and the worst-case expected length of the actual optimal confidence interval is no more than $4 \cdot \frac{\alpha_\gamma}{\sqrt{n}}(1 + O(1/n))$.*

The optimal solution of (4) is still difficult to solve. But if we replace the objective function by the sum of the squares of its two terms

$$4 \cdot \frac{\alpha_\gamma^2}{n+1} \cdot \frac{1}{n+1} \cdot \sum_{i=1}^{n+1} \frac{(1-U_i)z_i^2}{A(c_{(i)})} + \left(\frac{\sum_{i=1}^{n+1} z_i \cdot U_i}{n+1} \right)^2$$

the optimal solution can be computed efficiently. The optimization problem with the new objective function will give a 2-approximation of (4) because $a^2 + b^2 \leq (a+b)^2 \leq 2(a^2 + b^2)$ (see more details in the last paragraph of D.3). The characterization of the optimal solution is presented in the next section.

5.3 Mechanism

When the cost distribution is not available, our approach still uses the observed costs to approximate the underlying cost distribution, and at the same time, spends more money per data point in the earlier stages of the mechanism. But at each round i , we solve a slightly different optimization problem, whose objective function is the sum of the squares of the two terms in the objective function of (4). The new objective function will give a good approximation of the original problem. The mechanism is presented as Mechanism 2.

The characterization of the optimal solution U^i and A^i is as follows. Note that the mechanism does not use rule U^i but uses $\mathbb{1}(U^i(c) \geq \frac{1}{2})$ to purchase the i -th data point.

Theorem 5.1. *The optimal solution of (5) is as follows:*

$$\begin{aligned}
U_j^i &= \begin{cases} 0, & \text{if } \phi^i(c_{(j)}) < H \\ p \in (0, 1], & \text{if } \phi^i(c_{(j)}) = H \\ 1, & \text{if } \phi^i(c_{(j)}) > H \end{cases} \\
A^i(c_{(j)}) &= \min \left\{ 1, \frac{\lambda}{\sqrt{\phi^i(c_{(j)})}} \right\}
\end{aligned}$$

where p is a constant in $(0, 1]$, and $\phi^i(c)$ is the regularized virtual cost function (Definition 4.2) when the cost distribution is the uniform distribution over T_i , and λ is chosen such that the budget constraint is satisfied with equality. The value of λ and H can be computed using binary search over set T_i within time $O(\log |T_i|)$.

By the above theorem, the optimal mechanism ignores all the data points with regularized virtual costs above a threshold H , and purchases (with probability) all the data points below the threshold, which is very intuitive. The characterization of the optimal A remains the same as the unbiased case. We prove that the optimal H can be found by binary search because the objective function is a convex function of $\sum_{j=1}^i U_j$, when A is optimized after U is decided. The complete proof is in D.1.

Theorem 5.2. *Mechanism 2 (1) is truthful in expectation and individually rational; (2) satisfies the budget constraint $B = n\bar{B}$ in expectation; (3) and the for any cost distribution $\{c_1, \dots, c_n\}$, the worst-case expected length of the output confidence interval is no more than*

$$8\sqrt{10} \cdot L^* + \frac{2\sqrt{10}}{\sqrt{n}} + o\left(\frac{1}{\sqrt{n}}\right),$$

Mechanism 2: Mechanism for confidence interval

Input confidence level δ , total number of agents n and expected budget $B = n\bar{B}$.

for $i = 1, \dots, n$ **do**

- 1) Define set $T_i = \{c_1, \dots, c_{i-1}, \bar{C}\}$ for $i \geq 2$ and $T_i = \{\bar{C}\}$ for $i = 1$. We use $c_{(1)}, \dots, c_{(i)}$ to denote the elements in T_i ordered from smallest to largest.
- 2) Let A^i be the optimal solution of

$$\begin{aligned}
 A^i, U^i = \arg \min_{A, U} \max_{\mathbf{z} \in [0, 1]^i} & \quad 4 \cdot \frac{\alpha_\gamma^2}{n} \cdot \frac{1}{i} \sum_{j=1}^i \frac{(1 - U_j) z_j^2}{A(c_{(j)})} + \left(\frac{1}{i} \sum_{j=1}^i U_j \cdot z_j \right)^2 \\
 \text{s.t.} & \quad \sum_{j=1}^i (1 - U_j) \cdot A(c_{(j)}) \psi^i(c_{(j)}) \leq \frac{B}{16\sqrt{n/i}} \\
 & \quad (1 - U_j) \cdot A(c_{(j)}) \text{ is monotone non-increasing in } j \\
 & \quad 0 \leq A(c_{(j)}) \leq 1, \quad 0 \leq U_j \leq 1, \quad \forall j
 \end{aligned} \tag{5}$$

where $\psi^i(c)$ is the virtual cost function (Definition 3.1) when the cost distribution is the uniform distribution over T_i . The payment P^i is computed as in Lemma 3.2.

- 3) Ask agent i to report his cost c_i . Let $\mathbb{1}(U^i(c) \geq \frac{1}{2})$ be the rule that ignores data with cost c if $U^i(c) \geq \frac{1}{2}$, and never ignores the data if $U^i(c) < \frac{1}{2}$. Then the mechanism purchases agent i 's data z_i using the extended allocation and payment rule of

$$\left(A^i, \mathbb{1} \left(U^i(c) \geq \frac{1}{2} \right), P^i \right).$$

Let the collected data be \hat{x}_i .

end for

Output confidence interval

$$\left[\sum_{i=1}^n y_i/n - \frac{\alpha_\gamma}{\sqrt{n}} \cdot \hat{\sigma}, \quad \sum_{i=1}^n y_i/n + \frac{\sum_{i=1}^n \hat{U}_i}{n} + \frac{\alpha_\gamma}{\sqrt{n}} \cdot \hat{\sigma} \right],$$

where $\hat{\sigma}^2$ is the sample variance of y_1, \dots, y_n , and \hat{U}_i represents whether the i -th data point is ignored or not.

where L^* is the approximate worst-case expected length of the benchmark defined in Lemma 5.1.

Corollary 5.1. *The worst-case expected length of our mechanism's output confidence interval is no more than*

$$8\sqrt{10} \cdot OPT + \frac{2\sqrt{10} + 32\sqrt{10} \cdot \alpha_\gamma}{\sqrt{n}} + o\left(\frac{1}{\sqrt{n}}\right),$$

where OPT is the worst-case expected length of the optimal confidence interval estimator.

Discussion: As we show in the previous section, L^* is roughly $\frac{1}{\sqrt{n}} \cdot \left(2\alpha_\gamma \sqrt{\mathbb{E}\left[\frac{1-U_c^*}{A^*(c)}\right]} + \sqrt{n} \cdot \mathbb{E}[U_c^*] \right)$. If the problem is non-trivial, we should have $\frac{1}{\sqrt{n}}$ dominated by L^* .

The basic idea of the proof is the same as the unbiased case. We again define an intermediate mechanism (A', U', P') . It is more complicated to compare the mechanisms because we have one more bias term in the objective function. The complete proof can be found in D.3.

6 Discussions

In this work, we restrict our estimators to use only the collected data. When the data are correlated with the costs, the data analyst may gradually learn the cost-data correlation based on the collected pairs. This means that if a data holder arrives and reports his cost, the data analyst may form a prediction for his data based on his reported cost and the learned cost-data correlation, even if the data is not

collected. This leads to an interesting open question: Can the final estimation be improved if the data analyst makes use of such predicted data?

Allowing the data analyst to leverage on the cost-data correlation brings up an additional level of challenge when the data holders care about the privacy of their data. Such data holders may hesitate to report their costs, because reporting the cost itself reveals some information about his data. This makes it more challenging to achieve truthfulness in design an online mechanism.

Another open problem is whether it is possible to do better than the worst-case analysis. The optimality of our mechanism is based on the worst-case cost-data correlation. When the designer can gradually learn the cost-data correlation, is it possible to adjust the purchasing mechanism accordingly so that it achieves optimality with respect to the actual cost-data correlation?

More generally, it would be interesting to develop mechanisms for other more complicated statistical estimation tasks, such as hypothesis testing.

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A Concentration Bounds

A.1 Empirical Bernstein's Inequality

We show how to construct confidence interval using sample mean and sample variance of y_i 's.

Lemma A.1 (Maurer and Pontil [2009]). *Let X_1, \dots, X_n be independent random variables taking their values in $[0, M]$. Consider the sample expectation $\hat{\mu}$ and sample variance $\hat{\sigma}^2$ defined respectively by $\hat{\mu} = \sum_i X_i/n$ and $\hat{\sigma}^2 = \sum_{i=1}^n (X_i - \hat{\mu})^2/(n-1)$. Then with probability at least $1 - \gamma$,*

$$|\hat{\mu} - \mathbb{E}\hat{\mu}| \leq \hat{\sigma} \sqrt{\frac{2 \ln(4/\gamma)}{n}} + \frac{7M \ln(4/\gamma)}{3n}.$$

As we proved in the main text, $y_1 = \frac{\hat{x}_1}{A^1(c_1)}, \dots, y_n = \frac{\hat{x}_n}{A^n(c_n)}$ become independent when conditioning on a realization of arriving sequence $(\bar{\mathbf{c}}, \bar{\mathbf{z}})$, and when the estimator is unbiased, for any realization $(\bar{\mathbf{c}}, \bar{\mathbf{z}})$, $\mathbb{E}[\sum_i y_i/n] = \sum_i z_i/n = \mathbb{E}[z]$. Therefore by the above lemma, with probability at least $1 - \gamma$,

$$\left| \sum_i y_i/n - \mathbb{E}[z] \right| \leq \hat{\sigma} \sqrt{\frac{2 \ln(4/\gamma)}{n}} + \frac{7M \ln(4/\gamma)}{3n},$$

where $M = \max_i y_i$. Therefore the following interval is a confidence interval of $\mathbb{E}[z]$ with confidence level γ ,

$$\left[\sum_{i=1}^n y_i/n - \frac{\alpha_\gamma}{\sqrt{n}} \cdot \hat{\sigma}, \quad \sum_{i=1}^n y_i/n + \frac{\alpha_\gamma}{\sqrt{n}} \cdot \hat{\sigma} \right]$$

where $\alpha_\gamma = \sqrt{2 \ln(4/\gamma)} + \frac{7M \ln(4/\gamma)}{3\sqrt{n} \cdot \hat{\sigma}}$. Notice that the second term in α_γ , $\frac{7M \ln(4/\gamma)}{3\sqrt{n} \cdot \hat{\sigma}}$ is not a constant. But since we estimate $\hat{\sigma}^2$ with $\sum y_i^2/n$ (see Section 5.2) when computing the optimal confidence interval mechanism,

$$\frac{M}{\sqrt{n} \cdot \hat{\sigma}} = \frac{\max_i y_i}{\sqrt{n} \cdot \sqrt{\sum_{i=1}^n y_i^2/n}} \leq \frac{\max_i y_i}{\sqrt{\sum_{i=1}^n y_i^2}} \leq 1,$$

we can just assume that α_γ is a constant in the range of $\sqrt{2 \ln(4/\gamma)}$ to $\sqrt{2 \ln(4/\gamma)} + \frac{7 \ln(4/\gamma)}{3}$ when solving the optimization problem.

A.2 Estimation of Sample Standard Deviation

We show that when $0 \leq \mathbb{E}[y_i] \leq 1$, the difference between $\mathbb{E}[\hat{\sigma}]$ and $\sqrt{\mathbb{E}[\sum_{i=1}^n y_i^2/n]}$ is no more than $1 + O(1/n)$, where $\hat{\sigma}^2$ is the sample variance of y_1, \dots, y_n .

We first show that the difference between $\mathbb{E}[\hat{\sigma}^2]$ and $\frac{1}{n} \cdot \sum_{i=1}^n \mathbb{E}[y_i^2]$ is no more than 1. Because y_i 's are independent conditioned on c_1, \dots, c_n ,

$$\begin{aligned} \mathbb{E}[\hat{\sigma}^2 | c_1, \dots, c_n] &= \mathbb{E} \left[\sum_{i=1}^n \left(y_i - \sum_{i=1}^n y_i/n \right)^2 / (n-1) \middle| c_1, \dots, c_n \right] \\ &= \frac{1}{n-1} \cdot \mathbb{E} \left[\sum_{i=1}^n y_i^2 - \frac{2}{n} \sum_{i=1}^n y_i \sum_{i=1}^n y_i + n \cdot \left(\sum_{i=1}^n y_i/n \right)^2 \middle| c_1, \dots, c_n \right] \\ &= \frac{1}{n-1} \cdot \mathbb{E} \left[\sum_{i=1}^n y_i^2 - \frac{1}{n} \left(\sum_{i=1}^n y_i \right)^2 \middle| c_1, \dots, c_n \right] \\ &= \frac{1}{n-1} \cdot \mathbb{E} \left[\sum_{i=1}^n y_i^2 - \frac{1}{n} \sum_{i=1}^n y_i^2 - \frac{1}{n} \sum_{i=1}^n \sum_{j \neq i} y_i y_j \middle| c_1, \dots, c_n \right] \\ &= \frac{1}{n} \cdot \sum_{i=1}^n \mathbb{E}[y_i^2] - \frac{1}{n(n-1)} \sum_{i=1}^n \sum_{j \neq i} \mathbb{E}[y_i | c_1, \dots, c_n] \mathbb{E}[y_j | c_1, \dots, c_n] \\ &\in \left[\frac{1}{n} \cdot \sum_{i=1}^n \mathbb{E}[y_i^2] - 1, \quad \frac{1}{n} \cdot \sum_{i=1}^n \mathbb{E}[y_i^2] \right] \end{aligned}$$

Therefore

$$\sqrt{\frac{1}{n} \cdot \sum_{i=1}^n \mathbb{E}[y_i^2]} - 1 \leq \sqrt{\mathbb{E}[\widehat{\sigma}^2 | c_1, \dots, c_n]} \leq \sqrt{\frac{1}{n} \cdot \sum_{i=1}^n \mathbb{E}[y_i^2]}$$

for all c_1, \dots, c_n . It has been shown (e.g. in [Gurland and Tripathi \[1971\]](#)) that the difference between $\mathbb{E}[\widehat{\sigma} | c_1, \dots, c_n]$ and $\sqrt{\mathbb{E}[\widehat{\sigma}^2 | c_1, \dots, c_n]}$ decreases as sample size grows, dropping off as $O(1/n)$. Therefore

$$\sqrt{\frac{1}{n} \cdot \sum_{i=1}^n \mathbb{E}[y_i^2]} - 1 - O(1/n) \leq \mathbb{E}[\widehat{\sigma}] \leq \sqrt{\frac{1}{n} \cdot \sum_{i=1}^n \mathbb{E}[y_i^2]}.$$

The difference between $\mathbb{E}[\widehat{\sigma}]$ and $\sqrt{\mathbb{E}[\sum_{i=1}^n y_i^2/n]}$ is no more than $1 + O(1/n)$.

B Truthfulness and Virtual Cost Function

B.1 Extended Allocation Rule and Payment Rule

We apply the well-known Myerson's lemma to prove [Lemma 3.4](#).

Lemma B.1 ([Myerson and Satterthwaite \[1983\]](#)). *A survey mechanism $(A(c), P(c))$ defined on $c \in [0, \overline{C}]$ is truthful and pays 0 when $A(c) = 0$ if and only if*

- $A(c)$ is a monotone non-increasing function of c .
- $P(c) = c + \frac{1}{A(c)} \cdot \int_c^{\overline{C}} A(v) dv$.

Let (A^d, P^d) be a truthful and individually rational survey mechanism defined on a discrete cost set $\{c_1, \dots, c_m\}$ with $c_1 \leq \dots \leq c_m$. Then when we use the extended allocation rule and payment rule

$$A(c) = A^d(\lceil c \rceil), \quad P(c) = P^d(\lceil c \rceil), \quad \text{for all } c \in [0, c_m].$$

The allocation rule $A(c)$ must be monotone non-increasing because A^d must be monotone non-increasing. Suppose $\lceil c \rceil = c_i$, then

$$\begin{aligned} P(c) &= P^d(c_i) = c_i + \frac{1}{A(c_i)} \sum_{j=i+1}^m A(c_j)(c_j - c_{j-1}) \\ &= c + (c_i - c) + \frac{1}{A(c)} \sum_{j=i+1}^m A(c_j)(c_j - c_{j-1}) \\ &= c + (c_i - c) + \frac{1}{A(c)} \sum_{j=i+1}^m \int_{c_{j-1}}^{c_j} A(c_j) dv \\ &= c + \frac{1}{A(c)} \cdot \int_c^{c_m} A(v) dv, \end{aligned}$$

which is just the payment rule that satisfies the second condition of the Myerson's lemma. Therefore $(A(c), P(c))$ is truthful. It is also individually rational because we always have $P(c) \geq c$.

B.2 Virtual Cost Function

The following results on the virtual cost function ([Definition 3.1](#)) will be used in our proofs.

Lemma B.2. *Let $\psi(c)$ be the virtual cost function of the discrete uniform distribution over set $T = \{c_1, \dots, c_n\}$. Let T_i be an arbitrary subset of T with length i and let $\psi^i(c)$ be the virtual cost function of the discrete uniform distribution over set T_i . Then for any non-increasing allocation rule $A(c)$,*

$$\sum_{c \in T_i} A(c) \psi^i(c) \leq \sum_{c \in T} A(c) \psi(c).$$

Proof. Without loss of generality, assume $c_1 \leq \dots \leq c_n$ and $T_i = \{c_{p_1}, \dots, c_{p_i}\}$. Then according to the definition of $\psi(c)$ and $\psi^i(c)$, for all $1 \leq k \leq i$,

$$\begin{aligned} \sum_{j=p_{k-1}+1}^{p_k} \psi(c_j) &= \sum_{j=p_{k-1}+1}^{p_k} j \cdot c_j - (j-1)c_{j-1} \\ &= (p_k - p_{k-1}) \cdot c_{p_k} + p_{k-1} \cdot (c_{p_k} - c_{p_{k-1}}) \\ &\geq c_{p_k} + (k-1)(c_{p_k} - c_{p_{k-1}}) \\ &= \psi^i(c_{p_k}). \end{aligned}$$

since $p_k - p_{k-1} \geq 1$ and $p_{k-1} \geq k-1$.

For all $c_j \in T$, define $\lceil c_j \rceil$ to be the smallest cost in T_i that is greater than or equal to c_j .

$$\begin{aligned} \sum_{c \in T_i} A(c) \psi^i(c) &= \sum_{k=1}^i A(c_{p_k}) \psi^i(c_{p_k}) \\ &\leq \sum_{k=1}^i A(c_{p_k}) \left(\sum_{j=p_{k-1}+1}^{p_k} \psi(c_j) \right) \\ &= \sum_{j=1}^n A(\lceil c_j \rceil) \psi(c_j) \\ &\leq \sum_{j=1}^n A(c_j) \psi(c_j) \end{aligned}$$

since A is monotone non-increasing, □

C Proofs for Optimal Unbiased Estimator

In this section, we give the proofs for the optimal online unbiased estimator. In Section C.1, we characterize the optimal A^i by the use of the Lagrangian function and the KKT conditions. Since the objective function of our optimization problem is convex, we prove the optimality by constructing the dual variables that satisfy the KKT conditions with our primal solution. This is based on some properties of the regularized virtual cost function ϕ , which is presented at the beginning of the section. In Section C.2, we prove some lemmas that will be used in Section C.3 when we show the budget feasibility and the performance of our online mechanism.

C.1 Proof of Theorem 4.1

We first prove the following properties of the regularized virtual cost function ϕ when compared with the non-regularized virtual cost function ψ .

Lemma C.1. *For a discrete uniform distribution supported on $\{c_1, \dots, c_m\}$ with $c_1 \leq \dots \leq c_m$. Let ψ be its virtual cost function and ϕ be the regularized virtual cost function. For any $1 \leq i \leq m$, let I_i be the set of all the j that has $\phi(c_j) = \phi(c_i)$. Then*

1. $\phi(c)$ is a non-decreasing function of c .
2. $\sum_{j \in I_i} \phi(c_j) = \sum_{j \in I_i} \psi(c_j)$.
3. $\sum_{j=1}^i \phi(c_j) \leq \sum_{j=1}^i \psi(c_j)$ for all $1 \leq i \leq m$. If $\phi(c_i) \neq \phi(c_{i+1})$ then $\sum_{j=1}^i \phi(c_j) = \sum_{j=1}^i \psi(c_j)$.

Proof. Define $Avg(i, k)$ as the average of $\psi(i), \dots, \psi(k)$, i.e., $Avg(i, k) = \frac{1}{k-i+1} \sum_{j=i}^k \psi(c_j)$. Recall that the definition of $\phi(c_i)$ is as follows

$$\begin{aligned} \phi(c_i) &= \max\{\psi'(c_1), \dots, \psi'(c_i)\}, \\ \psi'(c_i) &= Avg(i, R_i), \end{aligned} \tag{6}$$

where R_i is the k to the right of i that minimizes $Avg(i, k)$, i.e.,

$$R_i = \arg \min_{k:k \geq i} Avg(i, k).$$

If there are multiple k 's that achieve the minimum, without loss of generality let R_i be the maximum of them.

We prove the following properties of the intervals $[i, R_i]$.

Claim C.1. *The intervals $[1, R_1], [2, R_2], \dots, [m, R_m]$ will not partially intersect, i.e., if $i < j$ and $j \leq R_i$ then $R_j \leq R_i$.*

Proof. We prove by contradiction. Suppose $i < j \leq R_i < R_j$, then since

$$\begin{aligned} Avg(j, R_i) &\leq Avg(i, j) \\ Avg(j, R_i) &\geq Avg(j, R_j) \end{aligned}$$

it satisfies that

$$Avg(i, R_i) \geq Avg(i, R_j),$$

which contradicts the definition of R_i . \square

Claim C.2. *For any j that is contained in $[i, R_i]$, it always holds that $Avg(j, R_j) \leq Avg(i, R_i)$, or equivalently $\psi'(c_j) \leq \psi'(c_i)$.*

Proof. First by Claim C.1, $[j, R_j] \subseteq [i, R_i]$. Then the claim can also be proved by contradiction. Assume there exists $i < j \leq R_j \leq R_i$ that has $Avg(j, R_j) > Avg(i, R_i)$. Then since R_j minimizes the average $Avg(j, k)$ over all $k \geq j$, we must also have $Avg(j, R_i) > Avg(i, R_i)$. But $Avg(j, R_i) > Avg(i, R_i)$ will immediately lead to $Avg(i, j-1) < Avg(i, R_i)$, as

$$\begin{aligned} &Avg(i, j-1) * (j-i) + Avg(j, R_i) * (R_i - j + 1) \\ &= Avg(i, R_i) * (j-i) + Avg(i, R_i) * (R_i - j + 1), \end{aligned}$$

which is contradictory to the definition of R_i that R_i minimizes $Avg(i, k)$ over all $k \geq i$. \square

Claim C.3. *If an interval $[i, R_i]$ is not contained in any other intervals. Then for any $j < i$, it holds that $Avg(j, R_j) < Avg(i, R_i)$, or equivalently $\psi'(j) < \psi'(i)$.*

Proof. By Claim C.1, $R_j < i$. We then prove by induction. First for $j = i-1$, by the definition of R_j , we must have $Avg(j, R_j) = Avg(j, j) < Avg(i, R_i)$, or equivalently $\psi'(j) < \psi'(i)$. Now suppose the claim holds for any $j \in [k+1, i-1]$, then for $j = k$, it must hold that $Avg(j, R_j) < \psi'(R_j+1) \leq \psi'(i)$. Again this is because R_j minimizes $Avg(j, R_j)$. \square

With the three results above, our lemma can be proved as follows:

(1): According to definition (6), ϕ is non-decreasing.

(2): Consider a fixed i , let L be the minimal j that has $R_j \geq i$, in other words let $[L, R_L]$ be the maximal interval that contains i . Then by Claim C.2 and Claim C.3,

$$\phi(c_i) = \max\{\psi'(c_1), \dots, \psi'(c_i)\} = \psi'(c_L) = Avg(L, R_L).$$

The same holds for any other $i \in [L, R_L]$. By Claim C.3, for any $j < L$,

$$\phi(c_j) = \max\{\psi'(c_1), \dots, \psi'(c_j)\} < \psi'(c_L).$$

And similarly for any $j > R_L$, $\psi'(c_j) > \psi'(c_L)$. Therefore $I_i = \{L, \dots, R_L\}$ and $\sum_{j \in I_i} \phi(c_j) = Avg(L, R_L) * (R_L - L + 1) = \sum_{j \in I_i} \psi(c_j)$.

(3): Consider a fixed i , let $I_i = \{L, \dots, R_L\}$ as proved in (2). If $\phi(c_i) \neq \phi(c_{i+1})$, then i must be the right end point of I_i . Then by summing over all I_j to the left of I_i , we get the equality $\sum_{j=1}^i \phi(c_j) = \sum_{j=1}^i \psi(c_j)$. Otherwise we have $\sum_{j=1}^{L-1} \phi(c_j) = \sum_{j=1}^{L-1} \psi(c_j)$ and $\sum_{j=L}^i \psi(c_j) = \text{Avg}(L, i) * (i - L + 1) \geq \text{Avg}(L, R_L) * (i - L + 1) = \sum_{j=L}^i \phi(c_j)$. \square

Now we prove the theorem: the solution of

$$\begin{aligned} A^i = \arg \min_A \max_{\mathbf{z} \in [0,1]^i} \sum_{k=1}^i \frac{z_k^2}{A(c(k))} \quad (7) \\ \text{s.t. } \sum_{k=1}^i A(c(k)) \psi^i(c(k)) \leq \frac{B}{4\sqrt{n/i}} \\ A(c(k)) \geq A(c(k+1)), \forall k \\ 0 \leq A(c(k)) \leq 1, \quad \forall k \end{aligned}$$

is $A^i(c_j) = \min \left\{ 1, \frac{\lambda}{\sqrt{\phi^i(c_j)}} \right\}$ for all $1 \leq j \leq i$, and λ is chosen such that the budget constraint is satisfied with equality

$$\sum_{j=1}^i A(c_j) \psi^i(c_j) = \frac{B}{4\sqrt{n/i}}$$

where ψ^i and ϕ^i are the non-regularized and regularized virtual cost function when the underlying cost distribution is the uniform distribution over T_i . The value of λ can be computed using binary search.

Proof. For simplicity, we write $A_k = A(c(k))$ and $A_k^i = A^i(c(k))$. Notice that the objective function is an increasing function of z_j for all j and thus the maximum is obtained when $z_j = 1$ for all j . Then the objective function of our minimization problem becomes $\sum_{j=1}^i \frac{1}{A_j}$. In the region $A_j > 0$ for all j , the objective function is convex. Therefore the KKT conditions are necessary and sufficient for optimality of the primal and dual variables (see [Boyd and Vandenberghe \[2004\]](#)). The Lagrangian function of the optimization problem is

$$\begin{aligned} L(A, \xi, \pi, \eta^0, \eta^1) = & \sum_{j=1}^i \frac{1}{A_j} \\ & + \xi \left(\sum_{j=1}^i A_j \psi^i(c_j) - \frac{B}{4\sqrt{n/i}} \right) \\ & + \sum_{j=1}^{i-1} \pi_j (A_{j+1} - A_j) \\ & + \sum_{j=1}^i \eta_j^0 A_j + \sum_{j=1}^i \eta_j^1 (A_j - 1). \end{aligned}$$

We prove that the optimal primal variables are $A^i(c) = \min \left\{ 1, \frac{\lambda}{\sqrt{\phi^i(c)}} \right\}$, where λ is chosen such that the budget constraint is satisfied with equality, and the optimal dual variables are

$$\begin{aligned} \xi &= \frac{1}{\lambda^2}, \\ \pi_j &= \pi_{j-1} + \xi (\psi^i(c_j) - \phi^i(c_j)), \quad (\text{Here we assume } \pi_0 = 0.) \\ \eta_j^0 &= 0 \text{ for all } j, \\ \eta_j^1 &= \begin{cases} 1 - \xi \cdot \phi^i(c_j), & \text{if } \xi \cdot \phi^i(c_j) < 1, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Primal feasibility: We first prove the primal feasibility.

1. By the definition of A^i , the budget constraint is satisfied with equality.
2. By (1) in Lemma C.1, $\phi^i(c)$ is a non-decreasing function of c . Then $A^i(c) = \min \left\{ 1, \frac{\lambda}{\sqrt{\phi^i(c)}} \right\}$ is a non-increasing function.
3. It is easy to verify that $0 \leq A^i(c) \leq 1$.

Dual feasibility: By (3) in Lemma C.1, it is easy to verify that all of the dual variables greater or equal to 0.

Stationarity: The partial derivative of $L(A, \xi, \pi, \eta^0, \eta^1)$ with respect to each A_j is

$$\begin{aligned} \frac{\partial L(A, \xi, \eta^0, \eta^1)}{\partial A_j} &= -\frac{1}{A_j^2} + \xi \cdot \psi^i(c_{(j)}) + \pi_{j-1} - \pi_j + \eta_j^0 + \eta_j^1 \\ &= -\max \{1, \xi \cdot \phi^i(c_{(j)})\} + \xi \cdot \psi^i(c_{(j)}) + \pi_{j-1} - \pi_j + \eta_j^1 \\ &= -\xi \cdot \phi^i(c_{(j)}) + \xi \cdot \psi^i(c_{(j)}) + \pi_{j-1} - \pi_j. \end{aligned}$$

By the definition of π_j , we have the above quantity equal to 0.

Complementary slackness:

1. The budget constraint in the primal is satisfied with equality.
2. For all $A_j^i \neq A_{j+1}^i$, we must have $\phi^i(c_{(j)}) \neq \phi^i(c_{(j+1)})$. Then by (3) in Lemma C.1, $\pi_j = 0$. Thus $\pi_j(A_{j+1}^i - A_j^i) = 0$ for all j .
3. We have $\eta_j^0 = 0$ for all j and $\eta_j^1 = 0$ for all $A_j^i = \frac{1}{\sqrt{\xi \cdot \phi^i(c_{(j)})}} < 1$.

As we have proved, the optimal allocation rule is monotone non-increasing. Since the sum $\sum_{j=1}^i A_j^i \psi^i(c_{(j)})$ is an increasing function of λ , we can perform binary search to find the right value of λ such that the sum equals the budget. Moreover, we can reduce the search space to $|T_i|$ by searching the critical point $c^* \in T_i$ that has $\frac{\lambda}{\sqrt{\phi^i(c)}} \geq 1$ for $c \leq c^*$ and $\frac{\lambda}{\sqrt{\phi^i(c)}} \leq 1$ for $c \geq c^*$.

This completes the proof of Lemma 4.1. \square

C.2 Optimal Mechanism for Adjacent Cost Sets

In this section, we show that the optimal solution A will not change a lot if we slightly modify the optimization problem (7), more specifically, if the set of costs contains one more element.

We first show that if the set of costs contains one more element, the regularized virtual costs function $\phi(c)$ of the uniform distribution over this set will change no more than a factor of 2, which is mainly because the virtual costs $\psi(c_i) = i \cdot c_i - (i-1)c_{i-1}$ change at most by a factor of 2.

Lemma C.2. *Let T_1 and T_2 be two costs sets that only differ in one element c_k . More specifically, suppose $T_2 = \{c_1, \dots, c_m\}$ with $c_1 \leq \dots \leq c_m$ and $T_1 = \{c_1, \dots, c_{k-1}, c_{k+1}, \dots, c_m\}$. Let $\phi_1(c)$ be the regularized virtual cost function (Definition 4.2) of the discrete uniform distribution over T_1 and let $\phi_2(c)$ be the regularized virtual cost function of the discrete uniform distribution over T_2 . Then it holds that for all $i \neq k$,*

$$\frac{1}{2} \cdot \phi_1(c_i) \leq \phi_2(c_i) \leq 2 \cdot \phi_1(c_i).$$

Proof. According to Definition 3.1, the (non-regularized) virtual costs are

$$\begin{aligned} \psi_1(c_1) &= c_1, \dots, \psi_1(c_{k-1}) = (k-1)c_{k-1} - (k-2)c_{k-2}, \\ \psi_1(c_{k+1}) &= k \cdot c_{k+1} - (k-1)c_{k-1}, \\ \psi_1(c_{k+2}) &= (k+1)c_{k+2} - k \cdot c_{k+1}, \dots, \psi_1(\bar{C}) = i \cdot \bar{C} - (i-1)c_i \end{aligned}$$

and

$$\begin{aligned}\psi_2(c_1) &= c_1, \dots, \psi_2(c_{k-1}) = (k-1)c_{k-1} - (k-2)c_{k-2}, \\ \psi_2(c_k) &= k \cdot c_k - (k-1)c_{k-1}, \psi_2(c_{k+1}) = (k+1) \cdot c_{k+1} - k \cdot c_k, \\ \psi_2(c_{k+2}) &= (k+2)c_{k+2} - (k+1) \cdot c_{k+1}, \dots, \psi_2(\bar{C}) = (i+1) \cdot \bar{C} - i \cdot c_i\end{aligned}$$

(i) For $i, j < k$, $Avg_1(i, j) = Avg_2(i, j)$.

(ii) For $Avg_2(i, j)$ that has $i < k$ and $j = k$, we have

$$\begin{aligned}Avg_1(i, k-1) - Avg_2(i, k) &= \frac{(k-1) \cdot c_{k-1} - (i-1)c_{i-1}}{k-i} - \frac{k \cdot c_k - (i-1)c_{i-1}}{k-i+1} \\ &= \frac{(i-1)(c_k - c_{i-1})}{(k-i)(k-i+1)} \\ &= \frac{(i-1)(c_k - c_{i-1})}{k-i+1} \cdot \frac{1}{k-i} \\ &\in [0, Avg_2(i, k)]\end{aligned}$$

Thus $\frac{1}{2} \cdot Avg_1(i, k-1) \leq Avg_2(i, k) \leq Avg_1(i, k-1)$.

(iii) For $i < k$ and $j \geq k+1$, we have $Avg_1(i, j) = \frac{(j-1)c_j - (i-1)c_{i-1}}{j-i}$ and $Avg_2(i, j) = \frac{j \cdot c_j - (i-1)c_{i-1}}{j-i+1}$. Then

$$Avg_1(i, j) - Avg_2(i, j) = \frac{(i-1)(c_j - c_{i-1})}{(j-i)(j-i+1)} \leq \frac{(i-1)(c_j - c_{i-1})}{j-i} \cdot \frac{1}{j-i+1} \in \left[0, Avg_1(i, j) \cdot \frac{1}{2}\right].$$

Thus $\frac{1}{2} Avg_1(i, j) \leq Avg_2(i, j) \leq Avg_1(i, j)$.

(iv) For $i, j \geq k+2$, we have $Avg_1(i, j) = \frac{(j-1)c_j - (i-2)c_{i-1}}{j-i+1}$ and $Avg_2(i, j) = \frac{j \cdot c_j - (i-1)c_{i-1}}{j-i+1}$. Then

$$Avg_2(i, j) - Avg_1(i, j) = \frac{c_j - c_{i-1}}{j-i+1} \leq \frac{c_j}{j-i+1} \leq \frac{(j-1)c_j - (i-2)c_{i-1}}{j-i+1} \in [0, Avg_1(i, j)].$$

Thus $Avg_1(i, j) \leq Avg_2(i, j) \leq 2 \cdot Avg_1(i, j)$.

According to the above four properties of the Avg function and the definition of $\psi'(c_i)$, we have

$$\frac{1}{2} \cdot \psi'_1(c_i) \leq \psi'_2(c_i) \leq 2\psi'_1(c_i), \text{ for } i < k \text{ and } i \geq k+2. \quad (8)$$

To compare $\psi'_1(c_i)$ and $\psi'_2(c_i)$ for the case $i = k+1$, we prove the follows

• $\max\{\psi'_2(\mathbf{c}_k), \psi'_2(\mathbf{c}_{k+1})\} \geq \frac{1}{2}\psi'_1(\mathbf{c}_{k+1})$: Observe that

$$\psi_2(c_k) + \psi_2(c_{k+1}) = (k+1)c_{k+1} - (k-1)c_{k-1} \geq k \cdot c_{k+1} - (k-1)c_{k-1} = \psi_1(c_{k+1}).$$

Therefore $Avg_2(k, k+1) \geq \frac{1}{2}Avg_1(k+1, k+1)$ and at least one of the $\psi_2(c_k)$ and $\psi_2(c_{k+1})$ must be no less than $\psi_1(c_{k+1})/2$. If $\psi_2(c_k) \geq \psi_1(c_{k+1})/2$, then

$$\begin{aligned}Avg_2(k, k) &\geq \frac{1}{2}Avg_1(k+1, k+1) \\ Avg_2(k, k+1) &\geq \frac{1}{2}Avg_1(k+1, k+1)\end{aligned}$$

In addition, by (iv) we have $Avg_2(k+2, j) \geq Avg_1(k+2, j)$ for all $j \geq k+2$, so

$$Avg_2(k, j) \geq \frac{1}{2}Avg_1(k+1, j)$$

for all $j \geq k+2$. Then

$$\psi'_2(c_k) = \min_{j \geq k} Avg_2(k, j) \geq \frac{1}{2} \min_{j \geq k+1} Avg_1(k+1, j) = \frac{1}{2}\psi'_1(c_{k+1}).$$

Similarly for the other case $\psi_2(c_{k+1}) \geq \psi_1(c_{k+1})/2$ we can get $\psi'_2(c_{k+1}) \geq \frac{1}{2}\psi'_1(c_{k+1})$.

- $\max\{\psi'_2(c_k), \psi'_2(c_{k+1})\} \leq 2\psi'_1(c_{k+1})$: Observe that

$$\begin{aligned}\psi_2(c_k) + \psi_2(c_{k+1}) &= (k+1) \cdot c_{k+1} - (k-1)c_{k-1} \\ &= c_{k+1} + (k \cdot c_{k+1} - (k-1)c_{k-1}) \\ &\leq 2(k \cdot c_{k+1} - (k-1)c_{k-1}) \\ &= 2\psi_1(c_{k+1}).\end{aligned}$$

Thus both $\psi_2(c_k)$ and $\psi_2(c_{k+1})$ is no greater than $2\psi_1(c_{k+1})$. Furthermore by (iv), $Avg_2(k+2, j) \leq 2Avg_1(k+2, j)$ for all $j \geq k+2$, it holds that

$$Avg_2(k, j) \leq 2Avg_1(k+1, j), \quad Avg_2(k+1, j) \leq 2Avg_1(k+1, j)$$

for all $j \geq k+1$. Thus

$$\begin{aligned}\psi'_2(c_k) &= \min_{j \geq k} Avg_2(k, j) \leq \min_{j \geq k+1} 2Avg_1(k+1, j) = 2\psi'_1(c_{k+1}) \\ \psi'_2(c_{k+1}) &= \min_{j \geq k+1} Avg_2(k+1, j) \leq \min_{j \geq k+1} 2Avg_1(k+1, j) = 2\psi'_1(c_{k+1})\end{aligned}$$

Combining the above two inequalities with (8),

$$\begin{aligned}\phi_2(c_i) &= \max_{j \leq i} \psi'_2(c_j) \leq 2 \cdot \max_{j \leq i} \psi'_1(c_j) = 2\phi_1(c_i), \text{ for all } i \neq k \\ \phi_1(c_i) &= \max_{j \leq i} \psi'_1(c_j) \leq 2 \cdot \max_{j \leq i} \psi'_2(c_j) = 2\phi_2(c_i), \text{ for all } i \neq k\end{aligned}$$

which completes the first part of the proof. \square

Based on the above analysis, we further prove the follows.

Lemma C.3. *Let T_1 and T_2 be two costs sets that have $T_2 = \{c_1, \dots, c_m\}$ with $c_1 \leq \dots \leq c_m$ and $T_1 = \{c_1, \dots, c_{k-1}, c_{k+1}, \dots, c_m\}$. Let $\phi_1(c)$ be the regularized virtual cost function (Definition 4.2) of the discrete uniform distribution over T_1 and let $\phi_2(c)$ be the regularized virtual cost function of the discrete uniform distribution over T_2 . Then it holds that for all number $K \geq 0$,*

$$\begin{aligned}\sum_{c \in T_1} \min \left\{ \phi_1(c), K\sqrt{\phi_1(c)} \right\} &\leq \sum_{c \in T_2} \min \left\{ 2\phi_2(c), K\sqrt{2\phi_2(c)} \right\} \\ \sum_{c \in T_2} \min \left\{ \phi_2(c), K\sqrt{\phi_2(c)} \right\} &\leq \sum_{c \in T_1} \min \left\{ 2\phi_1(c), 2K\sqrt{\phi_1(c)} \right\}.\end{aligned}$$

Proof. By Lemma C.2, for $i \neq k$, $\phi_1(c_i) \leq 2\phi_2(c_i)$. Thus the first inequality holds as follows,

$$\sum_{c \in T_1} \min \left\{ \phi_1(c), K\sqrt{\phi_1(c)} \right\} \leq \sum_{c \in T_2} \min \left\{ 2\phi_2(c), K\sqrt{2\phi_2(c)} \right\}.$$

By Lemma C.2, for $i \neq k$, $\phi_2(c_i) \leq 2\phi_1(c_i)$. In addition, we have

$$\begin{aligned}\psi_2(c_k) + \psi_2(c_{k+1}) &= (k+1)c_{k+1} - (k-1)c_{k-1} \\ &= c_{k+1} + k \cdot c_{k+1} - (k-1)c_{k-1} \\ &\leq 2(k \cdot c_{k+1} - (k-1)c_{k-1}) \\ &= 2\psi_1(c_{k+1}).\end{aligned}$$

By our definition of regularized virtual cost function, it always holds that

$$\sum_{c \in T_2} \min \left\{ \phi_2(c), K\sqrt{\phi_2(c)} \right\} \leq \sum_{c \in T_1} \min \left\{ 2\phi_1(c), 2K\sqrt{\phi_1(c)} \right\}.$$

\square

The following lemma will be used in our proof of the main theorem.

Lemma C.4. Let T_1 and T_2 be two costs sets that have $T_2 = \{c_1, \dots, c_k, \dots, c_m\}$ with $c_1 \leq \dots \leq c_m$ and $T_1 = \{c_1, \dots, c_{k-1}, c_{k+1}, \dots, c_m\}$. Let B be an arbitrary non-negative number. We use $M^{OPT}(T, B)$ to represent the optimal unbiased estimator mechanism defined in (3), when the cost set is T and the budget is B . Then define

- $(A_1, P_1) = M^{OPT}(T_1, B/2)$, i.e., (A_1, P_1) is the optimal mechanism when the cost set is T_1 and the budget is $B/2$.
- $(A_2, P_2) = M^{OPT}(T_2, B)$, i.e., (A_2, P_2) is the optimal mechanism when the cost set is T_2 and the budget is B .
- $(A_3, P_3) = M^{OPT}(T_2, B/4)$, i.e., (A_3, P_3) is the optimal mechanism when the cost set is T_2 and the budget is $B/4$.

Then we have

$$\begin{aligned} A_1(c_i) &\leq A_2(c_i) \text{ for all } i \geq k+1, & A_1(c_{k+1})P_1(c_{k+1}) &\leq A_2(c_{k+1})P_2(c_{k+1}), \\ A_1(c_i) &\geq A_3(c_i) \text{ for all } i \geq k+1, & A_1(c_{k+1})P_1(c_{k+1}) &\geq A_3(c_{k+1})P_3(c_{k+1}). \end{aligned}$$

Proof. Let $\phi_1(c)$ be the regularized virtual cost function (Definition 4.2) of the discrete uniform distribution over T_1 and let $\phi_2(c)$ be the regularized virtual cost function of the discrete uniform distribution over T_2 . Then according to Lemma 4.1,

$$A_i(c) = \min \left\{ 1, \frac{\lambda_i}{\sqrt{\phi_i(c)}} \right\}$$

and the value of λ_1 , λ_2 and λ_3 should satisfy

$$\begin{aligned} \sum_{c \in T_1} \min \left\{ \phi_1(c), \lambda_1 \sqrt{\phi_1(c)} \right\} &= B/2, \\ \sum_{c \in T_2} \min \left\{ \phi_2(c), \lambda_2 \sqrt{\phi_2(c)} \right\} &= B, \\ \sum_{c \in T_2} \min \left\{ \phi_2(c), \lambda_3 \sqrt{\phi_2(c)} \right\} &= B/4, \end{aligned}$$

We first compare A_1 and A_3 . By the first inequality of Lemma C.3,

$$\sum_{c \in T_1} \min \left\{ \phi_1(c), \sqrt{2} \cdot \lambda_3 \sqrt{\phi_1(c)} \right\} \leq \sum_{c \in T_2} \min \left\{ 2\phi_2(c), \sqrt{2} \cdot \lambda_3 \sqrt{2\phi_2(c)} \right\} \leq 2 \cdot B/4 = B/2.$$

Because the value of λ_1 is optimal when the cost set is T_1 and the budget is $B/2$, it should satisfy that $\lambda_1 \geq \sqrt{2} \cdot \lambda_3$. And since $\phi_1(c_i) \leq 2\phi_2(c_i)$ for all $i \geq k+1$ according to Lemma C.2, we get

$$A_3(c_i) = \min \left\{ 1, \frac{\lambda_3}{\sqrt{\phi_2(c_i)}} \right\} \leq \min \left\{ 1, \frac{\lambda_1/\sqrt{2}}{\sqrt{\phi_1(c_i)/2}} \right\} = A_1(c_i)$$

for all $i \geq k+1$. Similarly we have

$$A_1(c_i) = \min \left\{ 1, \frac{\lambda_1}{\sqrt{\phi_1(c_i)}} \right\} \leq \min \left\{ 1, \frac{\lambda_2/\sqrt{2}}{\sqrt{\phi_2(c_i)/2}} \right\} = A_2(c_i)$$

for all $i \geq k+1$.

The expected payment of the two allocation rules can be compared according to the definition in Lemma 3.2,

$$\begin{aligned} A_1(c_{k+1})P_1(c_{k+1}) &= A_1(c_{k+1})c_{k+1} + \int_{c_{k+1}}^{\bar{C}} A_1(v) dv \\ &\leq A_2(c_{k+1})c_{k+1} + \int_{c_{k+1}}^{\bar{C}} A_2(v) dv = A_2(c_{k+1})P_2(c_{k+1}). \end{aligned}$$

Here we use integrals to equivalently represent the payment rule as in Lemma B.1. \square

C.3 Proof of Theorem 4.2

We prove that when we use the following allocation rule

$$\begin{aligned}
A^i &= \arg \min_A \max_{\mathbf{z} \in [0,1]^i} \sum_{k=1}^i \frac{z_k^2}{A(c(k))} & (9) \\
\text{s.t. } & \sum_{k=1}^i A(c(k)) \psi^i(c(k)) \leq \frac{B}{4\sqrt{n/i}} \\
& A(c(k)) \geq A(c(k+1)), \forall k \\
& 0 \leq A(c(k)) \leq 1, \quad \forall k
\end{aligned}$$

at round i and output Horvitz-Thompson Estimator $S = \frac{1}{n} \sum_{i=1}^n y_i = \frac{1}{n} \sum_{i=1}^n \frac{\hat{x}_i}{A^i(c_i)}$ at last, the mechanism (1) is truthful and individually rational; (2) the expected total spending is no more than B ; (3) for any cost distribution $\{c_1, \dots, c_n\}$, the worst-case variance of the final estimator S is no more than roughly 16 times the benchmark defined in Definition 4.1. The worst case variance of the benchmark equals

$$\begin{aligned}
\text{Var}^*(A^*) &= \frac{1}{(n+1)^2} \min_A \max_{\mathbf{z} \in [0,1]^n} \sum_{i=1}^{n+1} \frac{z_i^2}{A(c_i)} - \sum_{i=1}^{n+1} z_i^2 & (10) \\
\text{s.t. } & \sum_{i=1}^{n+1} A(c_i) \psi(c_i) \leq B \\
& A(c_i) \geq A(c_{i+1}), \quad \forall 1 \leq i \leq n \\
& 0 \leq A(c) \leq 1, \quad \forall c
\end{aligned}$$

where we let $c_{n+1} = \bar{C}$ for notation simplicity.

To prove budget feasibility and the performance of the mechanism, we construct an intermediate mechanism (A', P') at each step i . This intermediate mechanism (A', P') is basically the same as (A^i, P^i) , but is “one-step-ahead”. Loosely speaking, (A', P') is the optimal mechanism when the same amount of budget is assigned at round i , but knows an additional piece of information, the value of c_i , beforehand. We compare (A', P') with (A^i, P^i) based on the results in Section C.2.

Notations: Before the proof, we define some notations. We use $\mathbb{E}[X|\{c_1, \dots, c_k\}]$ to represent the conditional expectation of random variable X conditioning on the event that the set of the first k data holders’ costs is $\{c_1, \dots, c_k\}$. We use $\mathbb{E}[X|c_1, \dots, c_k]$ to represent the conditional expectation of random variable X given that the sequence of the first k data holders’ costs is c_1, \dots, c_k , i.e., the first data holder has cost c_1 , the second has c_2 and so on so forth. Notations are the same for the conditional variance $\text{Var}(X|\{c_1, \dots, c_k\})$ and $\text{Var}(X|c_1, \dots, c_k)$. Unless otherwise stated, the randomness is taken over the random arriving order and the internal randomness of the mechanism.

C.3.1 Truthfulness and Individual Rationality

It is easy to see that the extended allocation rule and payment rule of a truthful mechanism is still truthful. The payment rule also guarantees individual rationality as $P^i(c)$ is always greater or equal to c by definition.

C.3.2 Expected Budget Feasibility

Suppose the costs of the population is $\{c_1, \dots, c_n\}$. The total expected spending of the mechanism is $\sum_{i=1}^n \mathbb{E}[A^i(c_i) \cdot P^i(c_i)]$. Consider a fixed round i . Let S_i be the set of first $i-1$ agents’ costs and define $T_i = S_i \cup \{\bar{C}\}$. Similarly let S_{i+1} be the set of first i agents’ costs and define $T_{i+1} = S_{i+1} \cup \{\bar{C}\}$. Then conditioning on the event that the set of first $i-1$ costs is S_i , the allocation rule A^i can be uniquely

decided, which is the solution of

$$\begin{aligned}
A^i &= \min_A \sum_{c \in T_i} \frac{1}{A(c)} \\
\text{s.t. } & \sum_{c \in T_i} A(c) \psi^i(c) \leq \frac{B}{4\sqrt{n/i}} \\
& A \text{ is monotone non-increasing} \\
& 0 \leq A(c) \leq 1, \quad \forall c \in T_i
\end{aligned}$$

Notice that $c_i \notin T_i$, so $A(c_i)$ is not a decision variable of the mathematical program defined above. The value of $A(c_i)$ is decided by Definition 3.2. Let $c_{(1)}, \dots, c_{(i)}$ be the first i costs in non-decreasing order and let $c_{(i+1)} = \bar{C}$. Suppose c_i is the k -th smallest cost, i.e., $c_i = c_{(k)}$. Then according to Definition 3.2,

$$A^i(c_i) = A^i(c_{(k)}) = A^i(c_{(k+1)})$$

where $c_{(k+1)}$ belongs to T_i . Now consider the following allocation rule A'

$$\begin{aligned}
A' &= \min_A \sum_{c \in T_{i+1}} \frac{1}{A(c)} \\
\text{s.t. } & \sum_{c \in T_{i+1}} A(c) \psi^{i+1}(c) \leq \frac{B}{2\sqrt{n/i}} \\
& A \text{ is monotone non-increasing} \\
& 0 \leq A(c) \leq 1, \quad \forall c \in T_{i+1}
\end{aligned}$$

Since T_i, T_{i+1} only differ by one element c_i , and A' uses twice the budget of A^i , according to Lemma C.4,

$$A^i(c_i) P^i(c_i) = A^i(c_{(k+1)}) P^i(c_{(k+1)}) \leq A'(c_{(k+1)}) P'(c_{(k+1)}).$$

Now assume the set of the first i costs is S_{i+1} . When the data holders come in random order, c_i is a random element chosen from S_{i+1} . Therefore c_i 's rank k should be uniformly distributed over $\{1, \dots, i\}$, and thus

$$\begin{aligned}
\mathbb{E}[A^i(c_i) \cdot P^i(c_i) | S_{i+1}] &\leq \mathbb{E}[A'(c_{(k+1)}) P'(c_{(k+1)}) | S_{i+1}] \\
&= \frac{1}{i} \cdot \sum_{j=1}^i A'(c_{(j+1)}) P'(c_{(j+1)}) \\
&\leq \frac{1}{2\sqrt{n \cdot i}} \cdot B.
\end{aligned}$$

Therefore the total spending of the mechanism is bounded as

$$\sum_{i=1}^n \mathbb{E}[A^i(c_i) \cdot P^i(c_i)] \leq \sum_{i=1}^n \frac{B}{2\sqrt{n \cdot i}} \leq B,$$

since $\sum_{i=1}^n \frac{1}{\sqrt{i}} \leq 2\sqrt{n}$.

C.3.3 Competitive Analysis

We first show that the variance of the final estimator $S = \frac{1}{n} \sum_{i=1}^n y_i$ can be upper bounded by the sum of the variances ‘‘occur at each round’’.

Lemma C.5. *For a population with costs $\{c_1, \dots, c_n\}$ and any cost-data distribution \mathcal{D} that is consistent with the costs, the variance of the output estimator of Mechanism 1 can be upper bounded as*

$$\text{Var}(S) \leq \frac{1}{n^2} \sum_{i=1}^n \mathbb{E}[y_i^2],$$

where $y_i = \frac{\hat{x}_i}{A^i(c_i)}$ is the re-weighted data.

Proof. According to the law of total variance,

$$\text{Var}(S) = \mathbb{E}[\text{Var}(S|c_1, \dots, c_n)] + \text{Var}(\mathbb{E}[S|c_1, \dots, c_n]),$$

Since the estimator is always unbiased, conditional mean $\mathbb{E}[S|c_1, \dots, c_n]$ always equals to the mean of n data points $\sum_{i=1}^n z_i/n$ for any order c_1, \dots, c_n . Because the variance of a constant is zero, the second term $\text{Var}(\mathbb{E}[S|c_1, \dots, c_n])$ equals 0. Therefore

$$\text{Var}(S) = \mathbb{E}[\text{Var}(S|c_1, \dots, c_n)].$$

Notice that conditioning on the cost sequence c_1, \dots, c_n , the allocation rules used at each round A^1, \dots, A^n will be fixed. Thus y_1, \dots, y_n become independent. So we have

$$\begin{aligned} \mathbb{E}[\text{Var}(S|c_1, \dots, c_n)] &= \mathbb{E}\left[\frac{1}{n^2} \cdot \text{Var}\left(\sum_{i=1}^n y_i \mid c_1, \dots, c_n\right)\right] \\ &= \frac{1}{n^2} \cdot \mathbb{E}\sum_{i=1}^n \text{Var}(y_i \mid c_1, \dots, c_n) \\ &\leq \frac{1}{n^2} \cdot \mathbb{E}\sum_{i=1}^n \mathbb{E}[y_i^2 \mid c_1, \dots, c_n] \\ &= \frac{1}{n^2} \sum_{i=1}^n \mathbb{E}[y_i^2] \end{aligned}$$

□

Let $y_i = \frac{\hat{x}_i}{A^i(c_i)}$ be the re-weighted observed data at round i . The following lemma compares the expectation $\mathbb{E}[y_i^2]$ with the worst-case variance of A^* .

Lemma C.6. *For any distribution \mathcal{D} that is consistent with the cost distribution $\{c_1, \dots, c_n\}$, when the data holders come in random order, Mechanism 1 has*

$$\mathbb{E}[y_i^2] \leq \frac{8 \cdot \sqrt{n}}{\sqrt{i}} \cdot \left(n \text{Var}^*(A^*) + 1 + \frac{1}{i} \cdot \frac{1}{A^*(\overline{C})} \right).$$

Proof. We fix a time step i . Let S_i be the set of first $i-1$ agents' costs and define $T_i = S_i \cup \{\overline{C}\}$. And similarly let S_{i+1} be the set of first i agents' costs and define $T_{i+1} = S_{i+1} \cup \{\overline{C}\}$. We first consider a fixed S_{i+1} and compare $\mathbb{E}[y_i^2|S_{i+1}]$ with $\text{Var}^*(A^*)$, where the randomness of $\mathbb{E}[y_i^2|S_{i+1}]$ is taken over the random arriving order and the internal randomness of the mechanism. We will define an intermediate allocation rule A' and compare it with both A^i and A^* .

Conditioning on the event that the set of first $i-1$ costs is S_i , the allocation rule A^i can be uniquely decided, which is the solution of

$$\begin{aligned} A^i &= \min_A \sum_{c \in T_i} \frac{1}{A(c)} \\ \text{s.t.} \quad &\sum_{c \in T_i} A(c) \psi^i(c) \leq \frac{B}{4\sqrt{n/i}} \\ &A \text{ is monotone non-increasing} \\ &0 \leq A(c) \leq 1, \quad \forall c \in T_i \end{aligned}$$

Notice that agent i 's cost $c_i \notin T_i$, so $A(c_i)$ is not a decision variable of the mathematical program defined above. The value of $A(c_i)$ is decided by Definition 3.2. Let $c_{(1)}, \dots, c_{(i)}$ be the first i costs in non-decreasing order and let $c_{(i+1)} = \overline{C}$. Suppose c_i is the k -th smallest cost, i.e., $c_i = c_{(k)}$. Then

$$A^i(c_i) = A^i(c_{(k)}) = A^i(c_{(k+1)})$$

where $c_{(k+1)}$ belongs to T_i . Now consider the following allocation rule A'

$$\begin{aligned} A' &= \min_A \sum_{c \in T_{i+1}} \frac{1}{A(c)} \\ \text{s.t.} \quad & \sum_{c \in T_{i+1}} A(c) \psi^{i+1}(c) \leq \frac{B}{8\sqrt{n/i}} \\ & A \text{ is monotone non-increasing} \\ & 0 \leq A(c) \leq 1, \quad \forall c \in T_{i+1} \end{aligned}$$

Comparing A' with A^* : We first compare A' with A^* by proving that $\frac{A^*}{8\sqrt{n/i}}$ is a feasible solution of the mathematical program that defines A' . Let $\psi(c)$ be the virtual cost function when the underlying cost distribution is the uniform distribution over $T = \{c_1, \dots, c_n, \bar{C}\}$. Then according to definition, the benchmark A^* satisfies the constraint $\sum_{c \in T} A^*(c) \psi(c) \leq B$. Combine this with Lemma B.2,

$$\sum_{c \in T_{i+1}} \frac{A^*}{8\sqrt{n/i}} \cdot \psi^{i+1}(c) \leq \sum_{c \in T} \frac{A^*}{8\sqrt{n/i}} \cdot \psi(c) \leq \frac{B}{8\sqrt{n/i}}.$$

Therefore $\frac{A^*}{8\sqrt{n/i}}$ is a feasible solution of the mathematical program that defines A' . Then since A' is the optimal solution, we have

$$\sum_{c \in T_{i+1}} \frac{1}{A'(c)} \leq \sum_{c \in T_{i+1}} \frac{8\sqrt{n/i}}{A^*(c)}. \quad (11)$$

Comparing A' with A^i : Now we compare $A^i(c_i) = A^i(c_{(k+1)})$ with $A'(c_{(k+1)})$. Since T_i and T_{i+1} only differs in one element $c_i = c_{(k)}$ and A' uses half of the budget of A^i , according to Lemma C.4,

$$A^i(c_{(k+1)}) \geq A'(c_{(k+1)}). \quad (12)$$

for any set of costs S_{i+1} of first i agents.

Conditioning on the event that the set of the first i costs is S_{i+1} , when the data holders come in random order, c_i is a random element chosen from $S_{i+1} = \{c_{(1)}, \dots, c_{(i)}\}$. Therefore c_i 's rank k should be uniformly distributed over $\{1, \dots, i\}$. Together with (12), we have

$$\mathbb{E}[y_i^2 | S_{i+1}] \leq \mathbb{E} \left[\frac{1}{A^i(c_{(k+1)})} \middle| S_{i+1} \right] \leq \mathbb{E} \left[\frac{1}{A'(c_{(k+1)})} \middle| S_{i+1} \right] = \frac{1}{i} \sum_{k=1}^i \frac{1}{A'(c_{(k+1)})}.$$

Then by equation (11),

$$\mathbb{E}[y_i^2 | S_{i+1}] \leq \frac{1}{i} \sum_{k=1}^i \frac{1}{A'(c_{(k+1)})} \leq \frac{1}{i} \sum_{c \in T_{i+1}} \frac{1}{A'(c)} \leq \frac{8 \cdot \sqrt{n/i}}{i} \sum_{c \in T_{i+1}} \frac{1}{A^*(c)}.$$

Now we are ready to compute $\mathbb{E}[y_i^2]$ by averaging $\mathbb{E}[y_i^2 | S_{i+1}]$ over random subset S_{i+1} (when the agents come in random order, S_{i+1} is a random subset of $\{c_1, \dots, c_n\}$ with length i).

$$\begin{aligned} \mathbb{E}[y_i^2] &= \mathbb{E}_{S_{i+1}} \mathbb{E}[y_i^2 | S_{i+1}] \\ &\leq \frac{8 \cdot \sqrt{n/i}}{i} \cdot \mathbb{E}_{S_{i+1}} \left[\sum_{c \in T_{i+1}} \frac{1}{A^*(c)} \right] \\ &= \frac{8 \cdot \sqrt{n/i}}{i} \cdot \mathbb{E}_{S_{i+1}} \left[\sum_{c \in S_{i+1}} \frac{1}{A^*(c)} + \frac{1}{A^*(\bar{C})} \right] \\ &= 8 \cdot \sqrt{n/i} \left(\frac{1}{n} \cdot \sum_{i=1}^n \frac{1}{A^*(c_i)} + \frac{1}{i} \cdot \frac{1}{A^*(\bar{C})} \right) \\ &\leq 8 \cdot \sqrt{n/i} \left(\frac{(n+1)^2}{n} \cdot \text{Var}^*(A^*) + 1 + \frac{1}{i} \cdot \frac{1}{A^*(\bar{C})} \right). \end{aligned}$$

□

Combining the two lemmas, for any joint distribution of cost and data, the variance of the output estimator of our mechanism satisfies

$$\begin{aligned}
\text{Var}(S) &\leq \frac{1}{n^2} \sum_{i=1}^n \mathbb{E}[y_i^2] \\
&\leq \frac{1}{n^2} \sum_{i=1}^n \frac{8 \cdot \sqrt{n}}{\sqrt{i}} \left(\frac{(n+1)^2}{n} \cdot \text{Var}^*(A^*) + 1 + \frac{1}{i} \cdot \frac{1}{A^*(\bar{C})} \right) \\
&= \frac{8\sqrt{n}}{n^2} \left(\frac{(n+1)^2}{n} \cdot \text{Var}^*(A^*) + 1 \right) \cdot \sum_{i=1}^n \frac{1}{\sqrt{i}} + \frac{8 \cdot \sqrt{n}}{n^2} \cdot \frac{1}{A^*(\bar{C})} \cdot \sum_{i=1}^n \frac{1}{i\sqrt{i}} \\
&\leq 16 \cdot \left(\frac{(n+1)^2}{n^2} \text{Var}^*(A^*) + \frac{1}{n} + \frac{\sqrt{n}}{n^2} \cdot \frac{1}{A^*(\bar{C})} \right)
\end{aligned}$$

since $\sum_{i=1}^n \frac{1}{\sqrt{i}} \leq 2\sqrt{n}$ and $\sum_{i=1}^n \frac{1}{i\sqrt{i}} \leq 2$.

This completes the proof of Theorem 4.2.

D Proofs for Optimal Confidence Interval

We then give the proofs for the optimal online confidence interval estimator. In Section D.1, we give the characterization of the optimal solution (A^i, U^i, P^i) and show how to compute it. In Section D.2, we prove some lemmas that will be used to compare our mechanism with the benchmark in Section D.3.

D.1 Proof of Theorem 5.1

We prove that the optimal solution of

$$\begin{aligned}
A^i, U^i &= \arg \min_{A, U} \quad 4 \cdot \frac{\alpha_\gamma^2}{n} \cdot \frac{1}{i} \sum_{j=1}^i \frac{1 - U_j}{A(c_{(j)})} + \left(\frac{1}{i} \sum_{j=1}^i U_j \right)^2 \tag{13} \\
&\text{s.t.} \quad \sum_{j=1}^i (1 - U_j) \cdot A(c_{(j)}) \psi^i(c_{(j)}) \leq \frac{B}{4\sqrt{n/i}} \\
&\quad (1 - U_j) \cdot A(c_{(j)}) \text{ is monotone non-increasing in } j \\
&\quad 0 \leq A(c_{(j)}) \leq 1, \quad 0 \leq U_j \leq 1, \quad \forall j
\end{aligned}$$

is as follows

$$U_j^i = \begin{cases} 0, & \text{if } \phi^i(c_{(j)}) < H \\ p \in (0, 1], & \text{if } \phi^i(c_{(j)}) = H \\ 1, & \text{if } \phi^i(c_{(j)}) > H \end{cases}$$

$$A^i(c_{(j)}) = \min \left\{ 1, \frac{\lambda}{\sqrt{\phi^i(c_{(j)})}} \right\}$$

λ is chosen such that $\sum_{j=1}^i (1 - U_j) \cdot A(c_{(j)}) \psi^i(c_{(j)}) = \frac{B}{4\sqrt{n/i}}$ and the optimal value of λ and H can be found using binary search.

The following lemma proves that the optimal solution should have U^i as described above for some H , and A^i should be non-increasing.

Lemma D.1. *Let U^i and A^i be the optimal solution of (13), then U_j^i should be non-decreasing in j and $A^i(c_{(j)})$ should be non-increasing in j . In addition, if $A^i(c_{(j)}) \neq A^i(c_{(j+1)})$, then at least one of U_j^i and U_{j+1}^i should be equal to 0 or 1.*

Proof. We first prove the monotonicity of U^i and A^i . Let U' and A' be a feasible solution. Suppose U' decreases at some position j , i.e., $U'_j > U'_{j+1}$. We prove that that U' cannot be optimal because it is not

the optimal solution of the following optimization problem (for simplicity we write $A_j = A(c_{(j)})$).

$$\begin{aligned}
& \min_{A_j, A_{j+1}, U_j, U_{j+1}} \frac{(1-U_j)}{A_j} + \frac{(1-U_{j+1})}{A_{j+1}} & (14) \\
& \text{s.t.} \quad (1-U_j)A_j = (1-U'_j)A'(c_{(j)}) \\
& \quad (1-U_{j+1})A_{j+1} = (1-U'_{j+1})A'(c_{(j+1)}) \\
& \quad (1-U_j) + (1-U_{j+1}) = (1-U'_j) + (1-U'_{j+1}) \\
& \quad 0 \leq U_j, U_{j+1}, A_j, A_{j+1} \leq 1.
\end{aligned}$$

It is easy to see that a feasible solution of the above problem is also a feasible solution of (13). And a strictly better solution than A', U' in the above problem will give a strictly better solution of (13). If we write the right-hand constant of the three equalities as D_1, D_2, D_3 and let $O_j = 1 - U_j$, then the objective function can be represented as a single variable function of O_j ,

$$\frac{(1-U_j)}{A_j} + \frac{(1-U_{j+1})}{A_{j+1}} = \frac{O_j^2}{D_1} + \frac{(D_3 - O_j)^2}{D_2}.$$

Its derivative with respect to O_j equals $\frac{2O_j}{D_1} - \frac{2(D_3 - O_j)}{D_2}$. The value of this derivative at point $O_j = 1 - U'_j$ is

$$2 \left(\frac{(1-U'_j)}{(1-U'_j)A'(c_{(j)})} - \frac{(1-U'_{j+1})}{(1-U'_{j+1})A'(c_{(j+1)})} \right).$$

Since A', U' is a feasible solution, it should satisfy the constraint $(1-U'_j)A'(c_{(j)}) \geq (1-U'_{j+1})A'(c_{(j+1)})$. Therefore if we have $1 \geq U'_j > U'_{j+1} \geq 0$, this derivative will be negative, which means that the objective value can be decreased by slightly decrease U'_j (it will still be feasible because $U'_j > U'_{j+1} \geq 0$). Therefore the optimal U^i must be monotone non-decreasing. The monotonicity of A^i can be proved by the same arguments, that is, to assume that $A'_j < A'_{j+1}$ for some j and show that it cannot be the optimal solution of the above optimization problem.

The same approach can be used to prove that the second part. Let U' and A' be an optimal solution with $A'(c_{(j)}) \neq A'(c_{(j+1)})$ and $U'_j, U'_{j+1} \in (0, 1)$ for some j . We again consider the optimization (14). Again the value of the derivative at point $O_j = 1 - U'_j$ is

$$2 \left(\frac{1}{A'(c_{(j)})} - \frac{1}{A'(c_{(j+1)})} \right).$$

Since an optimal allocation rule must be non-increasing, the value of this derivative is negative. By the same argument in the first part, U' cannot be optimal, which is contradictory. \square

Therefore the optimal $U_j^i \in (0, 1)$ only for j 's that have the same $A^i(c_{(j)})$. Based on this, we further prove the following lemma.

Lemma D.2. *Let l be the smallest number that has $U_l^i \in (0, 1)$ and let r be the largest number that has $U_r^i \in (0, 1)$. Then $\phi^i(c_{(l)}) = \phi^i(c_{(r)})$.*

Proof. Suppose to the contrary, $\phi^i(c_{(l)}) < \phi^i(c_{(r)})$. We construct another feasible solution that is strictly better than A^i, U^i . Define $I_l = \{k : \phi^i(c_{(k)}) = \phi^i(c_{(l)})\}$ and $I_r = \{k : \phi^i(c_{(k)}) = \phi^i(c_{(r)})\}$. I_l, I_r should both contain consecutive numbers. Let R_l be the right end point of I_l and let L_r be the left end point of I_r . First by definition of regularized virtual costs, $\phi^i(c_{(r)}) = \phi^i(c_{(L_r)}) = \min_{k:k \geq L_r} \text{Avg}(L_r, k)$, where Avg is the average function of ψ^i . Then

$$\text{Avg}(L_r, r) \geq \min_{k:k \geq L_r} \text{Avg}(L_r, k) = \phi^i(c_{(r)}).$$

Meanwhile according to Claim C.2 in the proof of Lemma C.1, we should have

$$\text{Avg}(l, R_l) \leq \text{Avg}(I_l) = \phi^i(c_{(l)}) < \phi^i(c_{(r)}).$$

In addition, if $R_l + 1 \leq L_r - 1$, then $\phi^i(c_{(j)}) < \phi^i(c_{(r)})$ for all $j \in [R_l + 1, L_r - 1]$, which means $\text{Avg}(R_l + 1, L_r - 1) < \phi^i(c_{(r)})$. Therefore we should have

$$\text{Avg}(l, L_r - 1) < \phi^i(c_{(r)}) \leq \text{Avg}(L_r, r). \quad (15)$$

Therefore we can construct a better solution by uniformly decreasing $U_l^i, \dots, U_{L_r-1}^i$ and uniformly increasing $U_{L_r}^i, \dots, U_r^i$ by a small amount, so that the total increase is equal to the total decrease. The value of the objective function $4 \cdot \frac{\alpha^2}{n} \cdot \frac{1}{i} \sum_{k=1}^i \frac{1-U_k^i}{A^i(c_{(k)})} + \left(\frac{1}{i} \sum_{k=1}^i U_k^i\right)^2$ remains the same, because (1) $\sum_{k=1}^i U_k^i$ does not change and (2) the value of $A^i(c_{(k)})$ is the same for all $k \in [l, r]$ according to Lemma D.1. But the total expected spending $\sum_{k=1}^i (1-U_k^i) \cdot A^i(c_{(k)}) \psi^i(c_{(k)})$ strictly decreases due to (15), which means that the objective value can be strictly improved. \square

So the lemma shows that $U_j^i \in (0, 1)$ only for j 's that have the same $\phi^i(c_{(j)})$. We then prove the other direction: for j 's that have the same $\phi^i(c_{(j)})$, there exists an optimal solution that has the same U_j^i for these j 's.

Lemma D.3. *There exists an optimal solution (U^i, A^i) such that for all (j, k) that has $\phi^i(c_{(j)}) = \phi^i(c_{(k)})$, $U_j^i = U_k^i$ and $A^i(c_{(j)}) = A^i(c_{(k)})$.*

Proof. Define P_H to be the set of j 's that have $\phi^i(c_{(j)}) = H$. Then the elements in P_H must be consecutive. Suppose $P_H = \{l, \dots, r\}$.

Let U^i, A^i be an optimal solution. We show that the following solution U', A' is also feasible and has objective value no larger than that of U^i, A^i :

$$U'_j = \frac{1}{r-l+1} \sum_{k=l}^r U_k^i, \quad \text{for all } j \in P_H$$

$$A'(c_{(j)}) = \frac{\sum_{k=l}^r (1-U_k^i) A^i(c_{(k)})}{\sum_{k=l}^r (1-U_k^i)}, \quad \text{for all } j \in P_H.$$

We first prove that U', A' is feasible. The monotonicity constraint and the range constraint are trivially satisfied. For the budget constraint on $\sum_{j=1}^i (1-U_j) \cdot A(c_{(j)}) \psi^i(c_{(j)})$, first observe that the sum $\sum_{k=l}^r (1-U'_k) A'(c_{(k)})$ remains the same as $\sum_{k=l}^r (1-U_k^i) A^i(c_{(k)})$,

$$\begin{aligned} \sum_{k=l}^r (1-U'_k) A'(c_{(k)}) &= \left(\sum_{k=l}^r 1 - U'_k \right) A'(c_{(l)}) \\ &= \left(\sum_{k=l}^r 1 - U_k^i \right) \frac{\sum_{k=l}^r (1-U_k^i) A^i(c_{(k)})}{\sum_{k=l}^r (1-U_k^i)} \\ &= \sum_{k=l}^r (1-U_k^i) A^i(c_{(k)}). \end{aligned}$$

By definition of regularized virtual costs, $Avg(l, r) = \phi^i(c_{(l)}) = \min_{k:k \geq l} Avg(l, k)$, where Avg is the average function of ψ^i . Thus

$$Avg(l, r) \leq Avg(l, k) \text{ for all } k \in [l, r].$$

And since $(1-U^i(c))A^i(c)$ must be non-increasing, by shifting $(1-U^i(c))A^i(c)$ to be constant within P_H , the total expected spending must only decrease,

$$\sum_{k=l}^r (1-U'_k) \cdot A'(c_{(k)}) \psi^i(c_{(k)}) \leq \sum_{k=l}^r (1-U_j^i) \cdot A^i(c_{(k)}) \psi^i(c_{(k)}).$$

Then we prove the optimality of U', A' . Because $\sum_{k=l}^r U'_k = \sum_{k=l}^r U_k^i$, the second term of the objective function remains the same. For the first term, since $f(x) = \frac{1}{x}$ is a convex function when $x > 0$,

by Jensen's inequality,

$$\begin{aligned}
\sum_{k=l}^r \frac{1-U'_k}{A'(c(k))} &= \left(\sum_{k=l}^r 1-U'_k \right) \frac{1}{A'(c(l))} \\
&= \left(\sum_{k=l}^r 1-U_k^i \right) \frac{1}{\sum_{k=l}^r \frac{1-U_k^i}{\sum_{k=l}^r 1-U_k^i} A^i(c(k))} \\
&\leq \left(\sum_{k=l}^r 1-U_k^i \right) \sum_{k=l}^r \frac{1-U_k^i}{\sum_{k=l}^r 1-U_k^i} \frac{1}{A^i(c(k))} \\
&= \sum_{k=l}^r (1-U_k^i) \frac{1}{A^i(c(k))}.
\end{aligned}$$

Therefore U', A' is no worse than U^i, A^i , and is thus optimal. \square

Combining the above three lemmas, there exists an optimal U^i as defined in the theorem. Next, we characterize the optimal A^i .

Lemma D.4. *If we fixed an optimal solution of $U: U_1^i, \dots, U_j^i$, which should have $U_j^i = U_k^i$ if $\phi^i(c(j)) = \phi^i(c(k))$ according to Lemma D.3, then the optimal solution of A should be equal to*

$$A^i(c(j)) = \min \left\{ 1, \frac{\lambda}{\sqrt{\phi^i(c(j))}} \right\},$$

where λ is chosen such that the budget constraint is satisfied with equality

$$\sum_{j=1}^i (1-U_j^i) \cdot A^i(c(j)) \psi^i(c(j)) = \frac{B}{4\sqrt{n/i}}.$$

Proof. When U is fixed as U_1^i, \dots, U_j^i , then the optimal A^i should be the solution of

$$\begin{aligned}
A^i &= \arg \min_A \sum_{j=1}^i \frac{(1-U_j^i)}{A(c(j))} \\
\text{s.t.} \quad &\sum_{j=1}^i (1-U_j^i) \cdot A(c(j)) \psi^i(c(j)) \leq \frac{B}{4\sqrt{n/i}} \\
&(1-U_j^i)A(c(j)) \text{ is monotone non-increasing in } j \\
&0 \leq A(c(j)) \leq 1, \quad \forall j
\end{aligned}$$

The Lagrangian function is (for simplicity we write $A_j = A(c(j))$)

$$\begin{aligned}
L(A, \xi, \pi, \eta^0, \eta^1) &= \sum_{j=1}^i \frac{1-U_j^i}{A_j} \\
&+ \xi \left(\sum_{j=1}^i (1-U_j^i) A_j \psi^i(c(j)) - \frac{B}{4\sqrt{n/i}} \right) \\
&+ \sum_{j=1}^i \pi_j \left((1-U_{j+1}^i) A_{j+1} - (1-U_j^i) A_j \right) \\
&+ \sum_{j=1}^i \eta_j^0 A_j + \sum_{j=1}^i \eta_j^1 (A_j - 1).
\end{aligned}$$

We prove that the optimal primal variables are

$$A_j^i = \min \left\{ 1, \frac{\lambda}{\sqrt{\phi^i(c(j))}} \right\},$$

where λ is chosen such that the budget constraint is satisfied with equality $\sum_{j=1}^i (1 - U_j^i) \cdot A_j^i \psi^i(c_{(j)}) = \frac{B}{4\sqrt{n/i}}$, and the optimal dual variables are

$$\begin{aligned}\xi &= \frac{1}{\lambda^2}, \\ \pi_j &= \pi_{j-1} + \xi (\psi^i(c_{(j)}) - \phi^i(c_{(j)})), \quad (\text{Here we assume } \pi_0 = 0.) \\ \eta_j^0 &= 0 \text{ for all } j, \\ \eta_j^1 &= \begin{cases} (1 - U_j^i) (1 - \xi \cdot \phi^i(c_{(j)})), & \text{if } \xi \cdot \phi^i(c_{(j)}) < 1, \\ 0, & \text{otherwise.} \end{cases}\end{aligned}$$

Primal feasibility: We first prove the primal feasibility.

1. By our definition of A^i , the budget constraint should be satisfied with equality.
2. By (1) in Lemma C.1, $\phi^i(c)$ is a non-decreasing function of c . Then $A^i(c) = \min \left\{ 1, \frac{\lambda}{\sqrt{\phi^i(c)}} \right\}$ is a non-increasing function. And since U_j^i is non-decreasing in j , $(1 - U_j^i)A_j^i$ is non-increasing in j .
3. It is easy to verify that $0 \leq A^i(c) \leq 1$.

Dual feasibility: By (3) in Lemma C.1, it is easy to verify that all of the dual variables greater or equal to 0.

Stationarity: The partial derivative of $L(A, \xi, \pi, \eta^0, \eta^1)$ with respect to each A_j is

$$\begin{aligned}& \frac{\partial L(A, \xi, \eta^0, \eta^1)}{\partial A_j} \\ &= -\frac{1 - U_j}{A_j^2} + \xi \cdot (1 - U_j) \psi^i(c_{(j)}) + (1 - U_j) \pi_{j-1} - (1 - U_j) \pi_j + \eta_j^0 + \eta_j^1 \\ &= -(1 - U_j) \max \{1, \xi \cdot \phi^i(c_{(j)})\} + \xi \cdot (1 - U_j) \cdot \psi^i(c_{(j)}) + (1 - U_j) \pi_{j-1} - (1 - U_j) \pi_j + \eta_j^1 \\ &= \xi(1 - U_j) \cdot (\psi^i(c_{(j)}) - \phi^i(c_{(j)})) + (1 - U_j) \pi_{j-1} - (1 - U_j) \pi_j \\ &= 0\end{aligned}$$

By the definition of π_j , we have the above quantity equal to 0.

Complementary slackness:

1. The budget constraint in the primal is satisfied with equality.
2. For all $A_j^i \neq A_{j+1}^i$, we must have $\phi^i(c_{(j)}) \neq \phi^i(c_{(j+1)})$. Then by (3) in Lemma C.1, $\pi_j = 0$. Since $U_j^i = U_{j+1}^i$ if $\phi^i(c_{(j)}) = \phi^i(c_{(j+1)})$ (which means $A_j^i = A_{j+1}^i$), $\pi_j ((1 - U_{j+1}^i)A_{j+1}^i - (1 - U_j^i)A_j^i) = 0$ for all j .
3. We have $\eta_j^0 = 0$ for all j and $\eta_j^1 = 0$ for all $A_j^i = \frac{1}{\sqrt{\xi \cdot \phi^i(c_{(j)})}} < 1$.

□

Therefore there must exist an optimal solution U^i, A^i as stated in the theorem.

By Lemma 4.1, the optimal value of λ can be found using binary search after U^i is decided. Define $M = \sum_{k=1}^i U_k^i$. Then when M is decided, we can immediately find a rule U^i that has the characterization in the theorem, and the optimal A^i can be computed as well. We now show how to find optimal M that minimizes

$$4 \cdot \frac{\alpha_\gamma^2}{n} \cdot \frac{1}{i} \sum_{j=1}^i \frac{1 - U_j}{A(c_{(j)})} + \left(\frac{M}{i} \right)^2$$

using binary search within time $O(\log |T_i|)$.

Let $g(M)$ be the optimal value of the first term $4 \cdot \frac{\alpha_\gamma^2}{n} \cdot \frac{1}{i} \sum_{j=1}^i \frac{1 - U_j}{A(c_{(j)})}$ when M is decided. Then the objective function is just $g(M) + \left(\frac{M}{i} \right)^2$. The second term $\left(\frac{M}{i} \right)^2$ is a convex function of M . We prove that $g(M)$ is also a convex function of M . Before the main proof, we first prove the following claim.

Claim D.1. Let U^i, A^i be an optimal solution as stated in the theorem. Then

$$\sum_{j=1}^i (1 - U_j^i) \cdot A^i(c_{(j)}) \phi^i(c_{(j)}) = \sum_{j=1}^i (1 - U_j^i) \cdot A^i(c_{(j)}) \psi^i(c_{(j)}) = \frac{B}{4\sqrt{n/i}}.$$

Proof. For each j , let I_j be the set of all the k that has $\phi^i(c_{(k)}) = \phi^i(c_{(j)})$. And let \mathcal{I} be the set of all different I_j , then according to (2) in Lemma C.1 and the definition of A^i ,

$$\begin{aligned} \sum_{j=1}^i (1 - U_j^i) A_j^i \phi^i(c_{(j)}) &= \sum_{I \in \mathcal{I}} \sum_{j \in I} (1 - U_j^i) A_j^i \phi^i(c_{(j)}) \\ &= \sum_{I \in \mathcal{I}} (1 - U_I^i) A_I^i \sum_{j \in I} \phi^i(c_{(j)}) \\ &= \sum_{I \in \mathcal{I}} (1 - U_I^i) A_I^i \sum_{j \in I} \psi^i(c_{(j)}) \\ &= \sum_{j=1}^i (1 - U_j^i) A_j^i \psi^i(c_{(j)}) = \frac{B}{4\sqrt{n/i}}, \end{aligned}$$

where A_I^i is the the same A_j^i for all $j \in I$, U_I^i is the same U_j^i for all $j \in I$. \square

Lemma D.5. The function $g(M)$ is a convex function of M . Furthermore, let A_M^i be an optimal allocation rule when $\sum_{j=1}^i U_j^i = M$ and let $c_{(r)}$ be the largest cost that is not ignored with probability 1. Then

$$\frac{\partial g(M)}{\partial M} = -8 \cdot \frac{\alpha_\gamma^2}{n} \cdot \frac{1}{i} \cdot \frac{1}{A_M^i(c_{(r)})}.$$

which is an non-decreasing function of M .

Proof. To simplify the notation, in the proof of this lemma, we use c_1, \dots, c_i instead of $c_{(1)}, \dots, c_{(i)}$ to represent the costs in T_i ordered from smallest to largest. To better illustrate how our objective value changes as M increases, we consider an equivalent optimization problem that is defined as follows:

$$\begin{aligned} A^i, M^i &= \arg \min_{A, M} 4 \cdot \frac{\alpha_\gamma^2}{n} \cdot \frac{1}{i} \int_0^{i-M} \frac{1}{A(x)} dx + \left(\frac{M}{i}\right)^2 \\ \text{s.t.} \quad &\int_0^{i-M} A(x) \cdot \phi^i(c_x) dx \leq \frac{B}{4\sqrt{n/i}} \\ &A \text{ is monotone non-increasing} \\ &0 \leq A(x) \leq 1, \quad \forall 0 \leq x \leq i \end{aligned} \tag{16}$$

where

$$c_x = c_{\lceil x \rceil}, \quad \forall 0 \leq x \leq i,$$

and thus $\phi^i(c_x) = \phi^i(c_{\lceil x \rceil})$. This optimization problem is equivalent to (13) because

1. For any feasible solution $(A(x), M)$ of the above optimization problem, we can find a feasible solution of (13) $(A(c_j), U_j)$ with better objective value by setting $A(c_j)$ equal to the mean of $A(x)$ for $x \in \{y : y \leq i - M \text{ and } \phi^i(c_y) = \phi^i(c_j)\}$, and setting U_j to be the mean of indicator function $\mathbb{1}(x \geq i - M)$ for $x \in \{y : \phi^i(c_y) = \phi^i(c_j)\}$. It can be verified that the budget constraint of (13) is still satisfied according to (2) in Lemma C.1. And the objective function only gets smaller because the mean of the reciprocals is no smaller than the reciprocal of the mean, as the reciprocal function is convex.
2. For any optimal solution $A(c_j), U_j$ of (13) that has the characterization as stated in the theorem, we can find a feasible solution $A(x), M$ of the above optimization problem with the same objective value by setting $M = \sum_j U_j$, $A(x) = A(c_{\lceil x \rceil})$.

Therefore by the characterization in Lemma D.4, if we fix a threshold M , there exists an optimal allocation rule, denoted by $A_M^i(x)$, which is equal to

$$A_M^i(x) = \min \left\{ 1, \frac{\lambda_M^i}{\sqrt{\phi^i(c_x)}} \right\}$$

for $0 \leq x \leq i - M$, and λ_M^i is chosen such that the budget constraint is satisfied with equality. By Claim D.1, the value of λ_M^i satisfies

$$\sum_{j=1}^i (1 - U_j^i) \cdot A^i(c_{(j)}) \phi^i(c_{(j)}) = \sum_{j=1}^i (1 - U_j^i) \cdot A^i(c_{(j)}) \psi^i(c_{(j)}) = \frac{B}{4\sqrt{n/i}},$$

or equivalently using $A_M^i(x)$ and M ,

$$\int_0^{i-M} A_M^i(x) \phi^i(c_x) dx = \int_0^{i-M} \min \left\{ \phi^i(c_x), \lambda_M^i \sqrt{\phi^i(c_x)} \right\} dx = \frac{B}{4\sqrt{n/i}}.$$

Recall that

$$g(M) = 4 \cdot \frac{\alpha_\gamma^2}{n} \cdot \frac{1}{i} \int_0^{i-M} \frac{1}{A_M^i(x)} dx.$$

Take derivative of the above function², by the Leibniz's rule for differentiation under the integral,

$$\begin{aligned} \frac{\partial g(M)}{\partial M} &= \frac{\partial}{\partial M} \left(4 \cdot \frac{\alpha_\gamma^2}{n} \cdot \frac{1}{i} \int_0^{i-M} \frac{1}{A_M^i(x)} dx \right) \\ &= 4 \cdot \frac{\alpha_\gamma^2}{n} \cdot \frac{1}{i} \left(-\frac{1}{A_M^i(i-M)} - \int_0^{i-M} \frac{1}{A_M^i(x)^2} \cdot \frac{\partial A_M^i(x)}{\partial M} dx \right). \end{aligned} \quad (17)$$

We denote the quantity inside the brackets by Q , then the derivative equals $4 \cdot \frac{\alpha_\gamma^2}{n} \cdot \frac{1}{i} \cdot Q$. Let t be the maximum of x that has $A_M^i(x) = 1$. Then for all $x \in [0, t]$, we should have $\frac{\partial A_M^i(x)}{\partial M} = 0$, because λ_M^i can only increase as M increase, which means $A_M^i(x)$ can only increase for $x \in [0, t]$, but they have already reached the upper bound 1, which means that $A_M^i(x)$ will stay the same as M increase for all $x \in [0, t]$. Then we can remove the part $0 \leq x < t$ of $\int_0^{i-M} -\frac{1}{A_M^i(x)^2} \cdot \frac{\partial A_M^i(x)}{\partial M} dx$ because it integrates to zero. So the term inside the brackets is equal to

$$\begin{aligned} Q &= -\frac{1}{A_M^i(i-M)} - \int_t^{i-M} \frac{1}{A_M^i(x)^2} \cdot \frac{\partial A_M^i(x)}{\partial M} dx \\ &= \frac{\partial}{\partial M} \left(\int_t^{i-M} \frac{1}{A_M^i(x)} dx \right) \\ &= \frac{\partial}{\partial M} \left(\int_t^{i-M} \frac{\sqrt{\phi^i(c_x)}}{\lambda_M^i} dx \right). \end{aligned} \quad (18)$$

To further simplify the derivative, we represent λ_M^i as a function of $\phi^i(c_x)$. Since λ_M^i is chosen such that the budget constraint is satisfied with equality, i.e.,

$$\int_0^{i-M} A_M^i(x) \phi^i(c_x) dx = \frac{B}{4\sqrt{n/i}}.$$

And since t is defined such that for $x \in [0, t]$, $A_M^i(x) = 1$; for $x \in (t, i - M]$, $A_M^i(x) = \frac{\lambda_M^i}{\sqrt{\phi^i(c_x)}}$, the above equality is equivalent to

$$\int_0^t \phi^i(c_x) + \int_t^{i-M} \lambda_M^i \sqrt{\phi^i(c_x)} dx = \frac{B}{4\sqrt{n/i}}.$$

²The function $g(M)$ may not be differentiable at integer points, but is semi-differentiable, WLOG we use right derivatives at those points.

Therefore the value of λ_M^i should satisfy

$$\lambda_M^i = \frac{\frac{B}{4\sqrt{n/i}} - \int_0^t \phi^i(c_x) dx}{\int_t^{i-M} \sqrt{\phi^i(c_x)} dx}.$$

Plug the above expression of λ_M^i into (18), we get

$$\begin{aligned} Q &= \frac{\partial}{\partial M} \left(\frac{\left(\int_t^{i-M} \sqrt{\phi^i(c_x)} dx \right)^2}{\frac{B}{4\sqrt{n/i}} - \int_0^t \phi^i(c_x) dx} \right) \\ &= \frac{1}{\frac{B}{4\sqrt{n/i}} - \int_0^t \phi^i(c_x) dx} \left(2 \cdot \int_t^{i-M} \sqrt{\phi^i(c_x)} dx \right) \left(-\sqrt{\phi^i(c_{i-M})} \right) \\ &= \frac{2}{\lambda_M^i} \left(-\sqrt{\phi^i(c_{i-M})} \right) \end{aligned}$$

by definition of λ_M^i . Plug it into (17),

$$\frac{\partial g(M)}{\partial M} = 4 \cdot \frac{\alpha_\gamma^2}{n} \cdot \frac{1}{i} \cdot \frac{2}{\lambda_M^i} \left(-\sqrt{\phi^i(c_{i-M})} \right).$$

Recall that M represents the amount of data that is ignored. When M increases, λ_M^i will increase, and $\sqrt{\phi^i(c_{i-M})}$ will decrease as the regularized virtual cost function is an increasing function. Therefore this partial derivative

$$8 \cdot \frac{\alpha_\gamma^2}{n} \cdot \frac{1}{i} \cdot \frac{1}{\lambda_M^i} \left(-\sqrt{\phi^i(c_{i-M})} \right) = -8 \cdot \frac{\alpha_\gamma^2}{n} \cdot \frac{1}{i} \cdot \frac{1}{A_M^i(i-M)}$$

is an increasing function of M . □

Then the optimal threshold

$$M^i = \arg \min_M g(M) + \left(\frac{M}{i} \right)^2.$$

should have the derivative greater than zero on its right, i.e., for $M > M^i$,

$$\frac{2M}{i^2} > 8 \cdot \frac{\alpha_\gamma^2}{n} \cdot \frac{1}{i} \cdot \frac{1}{A_M^i(i-M)},$$

and smaller than zero on the left, i.e., for $M < M^i$,

$$\frac{2M}{i^2} < 8 \cdot \frac{\alpha_\gamma^2}{n} \cdot \frac{1}{i} \cdot \frac{1}{A_M^i(i-M)}.$$

Thus the optimal M^i can be found by binary search.

D.2 Optimal Mechanism for Adjacent Cost Sets

The following lemmas will be used in the proof of the main theorem.

Lemma D.6. *Let T_1 and T_2 be two adjacent costs sets that have $T_2 = \{c_1, \dots, c_k, \dots, c_{i+1}\}$ with $c_1 \leq \dots \leq c_{i+1}$ and $T_1 = \{c_1, \dots, c_{k-1}, c_{k+1}, \dots, c_{i+1}\}$. Let B be an arbitrary non-negative number. We use $OPT(T, B)$ to represent the optimal confidence interval mechanism (at round i) defined in (5) when the cost set is T and the budget is B . Then define*

- $(A, U, P) = OPT(T_1, B/2)$, i.e., (A, U, P) is the optimal mechanism when the cost set is T_1 and the budget is $B/2$.
- $(A', U', P') = OPT(T_2, B)$, i.e., (A', U', P') is the optimal mechanism when the cost set is T_2 and the budget is B .

Then we have

$$(1 - U_{k+1})A(c_{k+1})P(c_{k+1}) \leq 2 \cdot (1 - U'_{k+1})A'(c_{k+1})P'(c_{k+1}).$$

Proof. The proof of this lemma is based on the proof of Lemma D.5.

We use $c_r \in T_1$ to represent the largest cost that has $U_r < 1$, and use $c_{r'} \in T_2$ to represent the largest cost that has $U'_{r'} < 1$. And define $M = \sum_{c \in T_1} U_c$ and $M' = \sum_{c \in T_2} U'_c$. To better illustrate the idea of our proof, we assume that the minimum point of our objective function is always achieved when its derivative with respect to M is equal to 0, which means that

$$\begin{aligned} a \cdot \frac{1}{A(c_r)} &= M \\ a \cdot \frac{1}{A'(c_{r'})} &= M' \end{aligned}$$

where $a = 4\alpha_\gamma^2 \cdot i/n$ is a constant (see the proof of Lemma D.5 for more details). The following analysis can be extended to the case when the derivative does not exist at the optimal point, but is greater than zero on the right and smaller than zero on the left.

First of all if $k > r'$, which means mechanism (A', U', P') never purchase c_k , as well as c_{k+1} . Then we can remove c_k from T_2 to get a new optimal mechanism $OPT(T_1, B)$. Then the new r' should only move to the left,³ which means we still have $k > r'$ and the new mechanism never purchase c_k . Because the budget of $(A, U, P) = OPT(T_1, B/2)$ is even smaller, it will not purchase c_k or c_{k+1} either. Therefore the inequality holds with both sides equal to zero.

Now suppose $k \leq r'$. To compare the two mechanisms, we define (A'', U'', P'') to be the optimal mechanism when (1) the cost set is T_1 and the budget is $B/2$ (which is the same as those of (A, U, P)) and (2) the total probability of ignoring the data points equals that of U' , i.e., $\sum_{c \in T_1} U''_c = \sum_{c \in T_2} U'_c$. Then we can apply Lemma C.4 to compare the non-zero parts of the two mechanisms. By the first line of the results in Lemma C.4,

$$A''(c_{r'}) \leq A'(c_{r'}),$$

and thus

$$a \cdot \frac{1}{A''(c_{r'})} \geq a \cdot \frac{1}{A'(c_{r'})} = M',$$

According to the proof of Lemma D.5, the optimal mechanism (A, U, P) must have $r \leq r'$ and $M \geq M'$. Then

$$A(c_r) = a/M \leq a/M' = A'(c_{r'}) \leq A'(c_r).$$

According to Lemma C.2, $\phi_2(c_r) \geq \frac{1}{2}\phi_1(c_r)$, together with the characterization of the optimal allocation rule in Theorem 5.1, (for the mechanism to be non-trivial, we assume $A(c_r) < 1$)

$$\lambda = A(c_r)\sqrt{\phi_1(c_r)} \leq A'(c_r)\sqrt{2\phi_2(c_r)} \leq \sqrt{2}\lambda'.$$

Again by Lemma C.2, $\phi_2(c_j) \leq 2\phi_1(c_j)$ for all $j \geq k+1$. Thus for all $j \geq k+1$,

$$A(c_j) = \min \left\{ 1, \frac{\lambda}{\sqrt{\phi_1(c_j)}} \right\} \leq \min \left\{ 1, \frac{\sqrt{2}\lambda'}{\sqrt{\phi_2(c_j)/2}} \right\} \leq 2 \min \left\{ 1, \frac{\lambda'}{\sqrt{\phi_2(c_j)}} \right\} = 2A'(c_j).$$

In addition, it should hold that $U^j \geq U'_j$ for all $j \geq k+1$.⁴ According to the definition in Lemma 3.2 (see $(1 - U(c))A(c)$ as an allocation rule here), the expected payment can be equivalently represented as an integral as in Lemma B.1,

$$\begin{aligned} (1 - U_{k+1})A(c_{k+1})P(c_{k+1}) &= (1 - U_{k+1})A(c_{k+1})c_{k+1} + \int_{c_{k+1}}^{\bar{C}} (1 - U_v)A(v) dv \\ &\leq 2(1 - U'_{k+1})A'(c_{k+1})c_{k+1} + 2 \int_{c_{k+1}}^{\bar{C}} (1 - U'_v)A'(v) dv \\ &= 2(1 - U'_{k+1})A'(c_{k+1})P'(c_{k+1}), \end{aligned}$$

which completes our proof. \square

³This is because (1) M' decreases and (2) $Avg(r' + 1, j)$ will only increase for all $j \geq r' + 1$.

⁴This is because (1) $M \geq M'$ and (2) $Avg(l, r)$ will only increase for all $l \leq k+1, r \geq k+1$ and $Avg(l, r)$ will only decrease for all $l, r > k+1$, so I_{k+1} will only extend to the right.

Lemma D.7. Let T_1 and T_2 be two adjacent costs sets that have $T_2 = \{c_1, \dots, c_k, \dots, c_{i+1}\}$ with $c_1 \leq \dots \leq c_{i+1}$ and $T_1 = \{c_1, \dots, c_{k-1}, c_{k+1}, \dots, c_{i+1}\}$. Let B be an arbitrary non-negative number. We use $OPT(T, B)$ to represent the optimal confidence interval mechanism (at round i) defined in (5) when the cost set is T and the budget is B . Then define

- $(A, U, P) = OPT(T_1, B)$, i.e., (A, U, P) is the optimal mechanism when the cost set is T_1 and the budget is B .
- $(A', U', P') = OPT(T_2, B/2)$, i.e., (A', U', P') is the optimal mechanism when the cost set is T_2 and the budget is $B/2$.

Define $M = \sum_{c \in T_1} U_c$ and $M' = \sum_{c \in T_2} U'_c$ and $W_{k+1} = \mathbb{1}(U_{k+1} \geq \frac{1}{2})$. Then we have

- $M \leq M'$.
- $4 \cdot \frac{\alpha_\gamma^2}{n} \cdot \frac{1 - W_{k+1}}{A(c_{k+1})} \leq 8 \cdot \frac{\alpha_\gamma^2}{n} \cdot \frac{1 - U'_{k+1}}{A'(c_{k+1})} + U'_{k+1} \cdot \frac{M'}{i}$.

Proof. The proof of this lemma is based on the proof of Lemma D.5.

Again we use $c_r \in T_1$ to represent the largest cost that has $U_r < 1$, and use $c_{r'} \in T_2$ to represent the largest cost that has $U'_{r'} < 1$. To better illustrate the idea of our proof, we assume that the minimum point of our objective function is achieved when its derivative with respect to M is equal to 0, which means that

$$a \cdot \frac{1}{A(c_r)} = M \quad (19)$$

$$a \cdot \frac{1}{A'(c_{r'})} = M' \quad (20)$$

where $a = 4\alpha_\gamma^2 \cdot i/n$ is a constant (see the proof of Lemma D.5 for more details). The following analysis can be extended to the case when the derivative does not exist at the optimal point, but is greater than zero on the right and smaller than zero on the left.

First consider the case $k+1 > r'$, which means $U'_{k+1} = 1$ and mechanism (A', U', P') never purchase c_{k+1} . If $k+1$ is also greater than r , which means that $W_{k+1} = 1$ and (A, U, P) never purchases c_{k+1} , then the inequality trivially holds with both sides equal to zero. If $k+1 \leq r$, then since $U'_{k+1} = 1$,

$$\begin{aligned} \frac{a}{i} \cdot \frac{1 - W_{k+1}}{A(c_{k+1})} &\leq \frac{a}{i} \cdot \frac{1}{A(c_r)} \\ &= \frac{M}{i} \\ &= \frac{a}{i} \cdot \frac{1 - U'_{k+1}}{A'(c_{k+1})} + U'_{k+1} \cdot \frac{M'}{i} \\ &\leq \frac{a}{i} \cdot \frac{1 - U'_{k+1}}{A'(c_{k+1})} + U'_{k+1} \cdot \frac{M'}{i}. \end{aligned}$$

Now suppose $k+1 \leq r'$. To compare the two mechanisms, we define (A'', U'', P'') to be the optimal mechanism when (1) the cost set is T_1 and the budget is B (which is the same as those of (A, U, P)) and (2) the total probability of ignoring the data points equals that of U' , i.e., $\sum_{c \in T_1} U''_c = \sum_{c \in T_2} U'_c$. Then we can apply Lemma C.4 to compare the non-zero parts of the two mechanisms. By the second line of the results in Lemma C.4,

$$A''(c_{r'}) \geq A'(c_{r'}),$$

and thus

$$a \cdot \frac{1}{A''(c_{r'})} \leq a \cdot \frac{1}{A'(c_{r'})} = M'.$$

According to the proof of Lemma D.5, the optimal mechanism (A, U, P) must have $M \leq M'$ and $r \geq r'$. Then by (19) and (20),

$$A(c_{r'}) \geq A(c_r) = a/M \geq a/M' = A'(c_{r'}).$$

According to Lemma C.2, $\phi_2(c_{r'}) \leq 2\phi_1(c_{r'})$, together with the characterization of the optimal allocation rule in Theorem 5.1, (for the mechanism to be non-trivial, we assume $A'(c_{r'}) < 1$)

$$\lambda \geq A(c_{r'})\sqrt{\phi_1(c_{r'})} \geq A'(c_{r'})\sqrt{\phi_2(c_{r'})/2} = \sqrt{1/2}\lambda'.$$

Again by Lemma C.2, $\phi_2(c_j) \geq \frac{1}{2}\phi_1(c_j)$ for all $j \geq k+1$. Thus for all $j \geq k+1$,

$$A(c_j) = \min \left\{ 1, \frac{\lambda}{\sqrt{\phi_1(c_j)}} \right\} \geq \min \left\{ 1, \frac{\sqrt{1/2}\lambda'}{\sqrt{2 \cdot \phi_2(c_j)}} \right\} \geq \frac{1}{2} \min \left\{ 1, \frac{\lambda'}{\sqrt{\phi_2(c_j)}} \right\} = \frac{1}{2}A'(c_j).$$

In addition, $\frac{1}{A(c_{k+1})} \leq \frac{1}{A(c_r)} = \frac{M}{a}$ due to the monotonicity constraint. Therefore

$$\begin{aligned} \frac{a}{i} \cdot \frac{1 - W_{k+1}}{A(c_{k+1})} &= \frac{a}{i} \cdot \frac{1 - U'_{k+1} + U'_{k+1}}{A(c_{k+1})} \\ &\leq \frac{a}{i} \cdot \frac{1 - U'_{k+1}}{A(c_{k+1})} + U'_{k+1} \cdot \frac{M}{i} \\ &\leq \frac{2a}{i} \cdot \frac{1 - U'_{k+1}}{A'(c_{k+1})} + U'_{k+1} \cdot \frac{M}{i} \\ &\leq \frac{2a}{i} \cdot \frac{1 - U'_{k+1}}{A'(c_{k+1})} + U'_{k+1} \cdot \frac{M'}{i}. \end{aligned}$$

which completes our proof. \square

D.3 Proof of Theorem 5.2

We now prove the main theorem for the online confidence interval mechanism. Define set $T_i = \{c_1, \dots, c_{i-1}, \overline{C}\}$. We use $c_{(1)}, \dots, c_{(i)}$ to denote the elements in T_i ordered from smallest to largest. Let

$$\begin{aligned} A^i, U^i = \arg \min_{A, U} \quad & 4 \cdot \frac{\alpha_\gamma^2}{n} \cdot \frac{1}{i} \sum_{j=1}^i \frac{(1 - U_j)}{A(c_{(j)})} + \left(\frac{1}{i} \sum_{j=1}^i U_j \right)^2 \\ \text{s.t.} \quad & \sum_{j=1}^i (1 - U_j) \cdot A(c_{(j)}) \psi^i(c_{(j)}) \leq \frac{B}{16\sqrt{n/i}} \\ & (1 - U_j) \cdot A(c_{(j)}) \text{ is monotone non-increasing in } j \\ & 0 \leq A(c_{(j)}) \leq 1, \quad 0 \leq U_j \leq 1, \quad \forall j \end{aligned}$$

We prove that when we use the extended allocation rule of $(A^i(c), \mathbb{1}(U^i(c) \geq \frac{1}{2}))$ at round i , and output confidence interval $\left[\sum_{i=1}^n y_i/n - \frac{\alpha_\gamma}{\sqrt{n}} \cdot \hat{\sigma}, \sum_{i=1}^n y_i/n + \frac{\hat{U}}{n} + \frac{\alpha_\gamma}{\sqrt{n}} \cdot \hat{\sigma} \right]$ at the end, where $\hat{U} = \sum_{i=1}^n \hat{U}_i$, our mechanism (1) is truthful in expectation and individually rational (2) satisfies the budget constraint B in expectation; (3) and the for any cost distribution $\{c_1, \dots, c_n\}$, the worst-case expected length of the output confidence interval is no more than

$$L \leq 8\sqrt{10} \cdot L^* + \frac{2\sqrt{10}}{\sqrt{n}} + o(1/\sqrt{n}),$$

where L^* is the approximate worst-case expected length of the benchmark A^*, U^* defined in Lemma 5.1,

$$\begin{aligned} L^* = \min_A \max_{\mathbf{z} \in [0,1]^n} \quad & 2 \cdot \frac{\alpha_\gamma}{\sqrt{n+1}} \cdot \sqrt{\frac{1}{n+1} \cdot \sum_{i=1}^{n+1} \frac{(1 - U_i) z_i^2}{A(c_{(i)})} + \frac{\sum_{i=1}^{n+1} z_i \cdot U_i}{n+1}} \\ \text{s.t.} \quad & \sum_{i=1}^{n+1} (1 - U_i) \cdot A(c_{(i)}) \psi(c_{(i)}) \leq B \\ & (1 - U_c)A(c) \text{ is monotone non-increasing in } c \\ & 0 \leq A(c) \leq 1, \quad 0 \leq U_c \leq 1, \quad \forall c \end{aligned} \tag{21}$$

For convenience we define

$$V^* = 4 \cdot \frac{\alpha_\gamma^2}{n} \cdot \frac{1}{n} \sum_{i=1}^n \frac{1 - U^*(c_i)}{A^*(c_i)}$$

so that $L^* \geq \frac{n}{n+1} \sqrt{V^*} + \frac{\sum_{i=1}^n U_i^*}{n}$, where $\sum_{i=1}^n U_i^*$ is the expected total number of data ignored by A^* .

The basic idea of the proof is the same as the unbiased case. We will again use an intermediate mechanism (A', U', P') to compare (A^i, U^i, P^i) with the benchmark at each round i . The difference of (A', U', P') and (A^i, U^i, P^i) is bounded using the results in the preceding section.

We use the same notations as in the proofs of the unbiased case C.3. The proofs of (1) in C.3 can be directly applied here, so we only prove (2) and (3).

D.3.1 Expected budget feasibility

The proof of budget feasibility is similar to the unbiased case. To bound $\sum_{i=1}^n \mathbb{E}[(1 - \widehat{U}_i)A^i(c_i)P^i(c_i)]$, we again define A' conditioning on a fixed $T_{i+1} = S_{i+1} \cup \{\overline{C}\}$.

$$\begin{aligned} A', U' = \arg \min_{A, U} & \quad 4 \cdot \frac{\alpha_\gamma^2}{n} \cdot \frac{1}{i} \sum_{c \in T_{i+1}} \frac{(1 - U_c)}{A(c)} + \left(\frac{1}{i} \sum_{c \in T_{i+1}} U_c \right)^2 \\ \text{s.t.} & \quad \sum_{c \in T_{i+1}} (1 - U_c) \cdot A(c) \psi^i(c) \leq \frac{B}{8\sqrt{n/i}} \\ & \quad (1 - U_c) \cdot A(c) \text{ is monotone non-increasing in } c \\ & \quad 0 \leq A(c) \leq 1, \quad 0 \leq U_c \leq 1, \quad \forall c \end{aligned}$$

Since T_i, T_{i+1} only differ by one element c_i , and A' uses twice the budget of A^i , according to Lemma D.6,

$$\begin{aligned} (1 - U^i(c_i))A^i(c_i)P^i(c_i) &= (1 - U^i(c_{(k+1)}))A^i(c_{(k+1)})P^i(c_{(k+1)}) \\ &\leq 2(1 - U'(c_{(k+1)}))A'(c_{(k+1)})P'(c_{(k+1)}). \end{aligned}$$

And thus

$$\begin{aligned} (1 - \mathbb{1}(U^i(c_{(k+1)}) \geq 1/2))A^i(c_{(k+1)})P^i(c_{(k+1)}) &\leq 2(1 - U^i(c_{(k+1)}))A^i(c_{(k+1)})P^i(c_{(k+1)}) \\ &\leq 4(1 - U'(c_{(k+1)}))A'(c_{(k+1)})P'(c_{(k+1)}). \end{aligned}$$

Now assume the set of the first i costs is S_{i+1} . When the data holders come in random order, c_i is a random element chosen from S_{i+1} . Therefore c_i 's rank k should be uniformly distributed over $\{1, \dots, i\}$, and thus

$$\begin{aligned} \mathbb{E}[(1 - \widehat{U}_i)A^i(c_i) \cdot P^i(c_i) | S_{i+1}] &= \frac{1}{i} \sum_{j=1}^i (1 - \mathbb{1}(U^i(c_{(j+1)}) \geq 1/2)) A^i(c_{(j+1)}) P^i(c_{(j+1)}) \\ &\leq \frac{1}{i} \sum_{j=1}^i 4(1 - U'(c_{(j+1)})) A'(c_{(j+1)}) P'(c_{(j+1)}) \\ &\leq \frac{4}{i} \cdot \sum_{c \in T_{i+1}} (1 - U'(c)) A'(c) P'(c) \\ &\leq \frac{1}{2\sqrt{n \cdot i}} \cdot B \end{aligned}$$

for all S_{i+1} . Therefore the total spending of the mechanism is bounded as

$$\sum_{i=1}^n \mathbb{E}[(1 - \widehat{U}_i)A^i(c_i) \cdot P^i(c_i)] \leq \sum_{i=1}^n \frac{B}{2\sqrt{n \cdot i}} \leq B,$$

since $\sum_{i=1}^n \frac{1}{\sqrt{i}} \leq 2\sqrt{n}$.

D.3.2 Competitive analysis

The expected length of the output confidence interval is equal to

$$L = 2 \cdot \frac{\alpha_\gamma}{\sqrt{n}} \cdot \mathbb{E}[\widehat{\sigma}] + \mathbb{E}[\widehat{U}]/n.$$

To compare it with L^* , we first upper bound the expectation of the sample variance $\mathbb{E}[\widehat{\sigma}^2]$ with $\mathbb{E}[y_i^2]$.

Lemma D.8. For any underlying distribution \mathcal{D} , our mechanism has

$$\mathbb{E}[\widehat{\sigma}^2] \leq \frac{1}{n} \sum_{i=1}^n \mathbb{E}[y_i^2],$$

where $\widehat{\sigma}^2$ is the sample variance of the re-weighted data $y_1 = \frac{\widehat{x}_1}{A^1(c_1)}, \dots, y_n = \frac{\widehat{x}_n}{A^n(c_n)}$.

Proof. Because y_i 's are independent conditioned on c_1, \dots, c_n ,

$$\begin{aligned} \mathbb{E}[\widehat{\sigma}^2 | c_1, \dots, c_n] &= \mathbb{E} \left[\sum_{i=1}^n \left(y_i - \sum_{i=1}^n y_i / n \right)^2 / (n-1) \middle| c_1, \dots, c_n \right] \\ &= \frac{1}{n-1} \cdot \mathbb{E} \left[\sum_{i=1}^n y_i^2 - \frac{2}{n} \sum_{i=1}^n y_i \sum_{i=1}^n y_i + n \cdot \left(\sum_{i=1}^n y_i / n \right)^2 \middle| c_1, \dots, c_n \right] \\ &= \frac{1}{n-1} \cdot \mathbb{E} \left[\sum_{i=1}^n y_i^2 - \frac{1}{n} \left(\sum_{i=1}^n y_i \right)^2 \middle| c_1, \dots, c_n \right] \\ &= \frac{1}{n-1} \cdot \mathbb{E} \left[\sum_{i=1}^n y_i^2 - \frac{1}{n} \sum_{i=1}^n y_i^2 - \frac{1}{n} \sum_{i=1}^n \sum_{j \neq i} y_i y_j \middle| c_1, \dots, c_n \right] \\ &= \frac{1}{n} \cdot \sum_{i=1}^n \mathbb{E}[y_i^2] - \frac{1}{n(n-1)} \sum_{i=1}^n \sum_{j \neq i} \mathbb{E}[y_i | c_1, \dots, c_n] \mathbb{E}[y_j | c_1, \dots, c_n] \\ &\leq \frac{1}{n} \cdot \sum_{i=1}^n \mathbb{E}[y_i^2] \end{aligned}$$

□

We then prove that the sum of the squares of the two terms in L can be bounded using V^* and $(U^*/n)^2$. We first prove the following lemma:

Lemma D.9. Let S_{i+1} be the set of first i agents' costs. When the agents come in random order, S_{i+1} is a random subset of $\{c_1, \dots, c_n\}$ with length i . Then it satisfies that

$$4 \cdot \frac{\alpha_\gamma^2}{n} \cdot \mathbb{E}[\widehat{\sigma}^2] + \left(\mathbb{E}[\widehat{U}] / n \right)^2 \leq \frac{1}{n} \sum_{i=1}^n \mathbb{E}_{S_{i+1}} \left[4 \cdot \frac{\alpha_\gamma^2}{n} \cdot \mathbb{E}[y_i^2 | S_{i+1}] + \mathbb{E}[\widehat{U}_i | S_{i+1}]^2 \right].$$

Proof. First by Jensen's inequality,

$$4 \cdot \frac{\alpha_\gamma^2}{n} \cdot \mathbb{E}[\widehat{\sigma}^2] + (\mathbb{E}[\widehat{U}] / n)^2 \leq 4 \cdot \frac{\alpha_\gamma^2}{n} \cdot \mathbb{E}[\widehat{\sigma}^2] + \left(\sum_{i=1}^n \mathbb{E}[\widehat{U}_i] / n \right)^2 \leq 4 \cdot \frac{\alpha_\gamma^2}{n} \cdot \mathbb{E}[\widehat{\sigma}^2] + \frac{1}{n} \sum_{i=1}^n \mathbb{E}[\widehat{U}_i]^2.$$

As we prove in Lemma D.8, $\mathbb{E}[\widehat{\sigma}^2] \leq \frac{1}{n} \sum_{i=1}^n \mathbb{E}[y_i^2]$, so we further have

$$\begin{aligned} 4 \cdot \frac{\alpha_\gamma^2}{n} \cdot \mathbb{E}[\widehat{\sigma}^2] + \left(\mathbb{E}[\widehat{U}] / n \right)^2 &\leq 4 \cdot \frac{\alpha_\gamma^2}{n} \cdot \frac{1}{n} \sum_{i=1}^n \mathbb{E}[y_i^2] + \frac{1}{n} \sum_{i=1}^n \mathbb{E}[\widehat{U}_i]^2 \\ &\leq 4 \cdot \frac{\alpha_\gamma^2}{n} \cdot \frac{1}{n} \sum_{i=1}^n \mathbb{E}_{S_{i+1}} \mathbb{E}[y_i^2 | S_{i+1}] + \frac{1}{n} \sum_{i=1}^n \mathbb{E}_{S_{i+1}} \mathbb{E}[\widehat{U}_i | S_{i+1}]^2 \\ &= \frac{1}{n} \sum_{i=1}^n \mathbb{E}_{S_{i+1}} \left[4 \cdot \frac{\alpha_\gamma^2}{n} \cdot \mathbb{E}[y_i^2 | S_{i+1}] + \mathbb{E}[\widehat{U}_i | S_{i+1}]^2 \right]. \end{aligned}$$

□

Then the sum of the squares of the two terms in L can be bounded as follows:

Lemma D.10. For any underlying distribution \mathcal{D} , our mechanism has

$$\begin{aligned} & 4 \cdot \frac{\alpha_\gamma^2}{n} \cdot \mathbb{E}[\hat{\sigma}^2] + \left(\mathbb{E}[\hat{U}] / n \right)^2 \\ & \leq 320 \cdot V^* + 20 \left(\frac{\sum_{i=1}^n U_i^*}{n} \right)^2 + 10 \left(\frac{1 + \ln n}{n} \right)^2 + \frac{2}{n} \left(640 \cdot \frac{\alpha_\gamma^2}{\sqrt{n}} \cdot \frac{1 - U_{\overline{C}}^*}{A^*(\overline{C})} + 10 (U_{\overline{C}}^*)^2 \right). \end{aligned}$$

where $\hat{\sigma}^2$ is the sample variance of the re-weighted data $y_1 = \frac{\hat{x}_1}{A^1(c_1)}, \dots, y_n = \frac{\hat{x}_n}{A^n(c_n)}$, and \hat{U} is the number of data points ignored by the mechanism, i.e., $\hat{U} = \sum_{i=1}^n \hat{U}_i$.

Proof. Let S_{i+1} be the set of first i agents' costs. When the agents come in random order, S_{i+1} is a random subset of $\{c_1, \dots, c_n\}$ with length i . We first consider a fixed S_{i+1} and bound $4 \cdot \frac{\alpha_\gamma^2}{n} \cdot \mathbb{E}[y_i^2 | S_{i+1}] + \mathbb{E}[\hat{U}_i | S_{i+1}]^2$. Let $S_i \subseteq S_{i+1}$ be the set of first $i-1$ costs is S_i , the allocation rule A^i is uniquely decided as in (13) where $T_i = S_i \cup \{\overline{C}\}$. Again if $c_i = c_{(k)}$, we will have $A^i(c_i) = A^i(c_{(k+1)})$.

Let $T_{i+1} = S_{i+1} \cup \{\overline{C}\}$. We define the following intermediate allocation rule A', U'

$$\begin{aligned} A', U' = \arg \min_{A, U} & \quad 4 \cdot \frac{\alpha_\gamma^2}{n} \cdot \frac{1}{i} \sum_{c \in T_{i+1}} \frac{(1 - U_c)}{A(c)} + \left(\frac{1}{i} \sum_{c \in T_{i+1}} U_c \right)^2 \\ \text{s.t.} & \quad \sum_{c \in T_{i+1}} (1 - U_c) \cdot A(c) \psi^i(c) \leq \frac{B}{32\sqrt{n/i}} \\ & \quad (1 - U_c) \cdot A(c) \text{ is monotone non-increasing in } c \\ & \quad 0 \leq A(c) \leq 1, \quad 0 \leq U_c \leq 1, \quad \forall c \end{aligned} \quad (22)$$

Claim D.2 (Compare $(A^i, \mathbb{1}(U^i \geq \frac{1}{2}))$ with (A', U')). Define $M' = \sum_{c \in T_{i+1}} U'_c$, then

$$4 \cdot \frac{\alpha_\gamma^2}{n} \cdot \mathbb{E}[y_i^2 | S_{i+1}] + \mathbb{E}[\hat{U}_i | S_{i+1}]^2 \leq 5 \left(4 \cdot \frac{\alpha_\gamma^2}{n} \cdot \frac{1}{i} \sum_{c \in T_{i+1}} \frac{1 - U'_c}{A'(c)} + \left(\frac{M'}{i} \right)^2 \right).$$

Proof. When the data holders come in random order, the set of first $i-1$ costs S_i is a random subset of S_{i+1} . Therefore c_i 's rank k is uniformly distributed over $\{1, \dots, i\}$. Let $A^{i,k}, U^{i,k}$ be the optimal solution in round i when the i -th cost c_i has rank k , i.e., $c_i = c_{(k)}$. Then

$$\begin{aligned} & 4 \cdot \frac{\alpha_\gamma^2}{n} \cdot \mathbb{E}[y_i^2 | S_{i+1}] + \mathbb{E}[\hat{U}_i | S_{i+1}]^2 \\ & = 4 \cdot \frac{\alpha_\gamma^2}{n} \left(\frac{1}{i} \sum_{k=1}^i \frac{1 - \mathbb{1}(U_{k+1}^{i,k} \geq \frac{1}{2})}{A^{i,k}(c_{(k+1)})} \right) + \left(\frac{1}{i} \sum_{k=1}^i \mathbb{1}(U_{k+1}^{i,k} \geq \frac{1}{2}) \right)^2. \end{aligned}$$

Denote $\mathbb{1}(U_{k+1}^{i,k} \geq \frac{1}{2})$ by $W_{k+1}^{i,k}$, the above equality becomes

$$4 \cdot \frac{\alpha_\gamma^2}{n} \cdot \mathbb{E}[y_i^2 | S_{i+1}] + \mathbb{E}[\hat{U}_i | S_{i+1}]^2 = 4 \cdot \frac{\alpha_\gamma^2}{n} \left(\frac{1}{i} \sum_{k=1}^i \frac{1 - W_{k+1}^{i,k}}{A^{i,k}(c_{(k+1)})} \right) + \left(\frac{1}{i} \sum_{k=1}^i W_{k+1}^{i,k} \right)^2. \quad (23)$$

To compare it with (A', U') , we first claim the follows

Claim D.3. Let $M' = \sum_{c \in T_{i+1}} U'_c$, then

$$\frac{1}{i} \sum_{k=1}^i W_{k+1}^{i,k} = \frac{1}{i} \sum_{k=1}^i \mathbb{1}(U_{k+1}^{i,k} \geq \frac{1}{2}) \leq 2M'/i.$$

Proof. Define

$$M^{i,k} = U_1^{i,k} + \dots + U_{k-1}^{i,k} + U_{k+1}^{i,k} + \dots + U_i^{i,k}.$$

By Lemma D.7, $M^{i,k} \leq M'$ for every $k \in [i]$, which means

$$U_1^{i,k} + \dots + U_{k-1}^{i,k} + U_{k+1}^{i,k} + \dots + U_i^{i,k} \leq M',$$

so each $U^{i,k}$ has at most $2M'$ entries that are no less than $\frac{1}{2}$. And since $U^{i,k}$ is monotone non-decreasing, i.e., $U_1^{i,k} \leq \dots \leq U_{k-1}^{i,k} \leq U_{k+1}^{i,k} \leq \dots \leq U_i^{i,k}$, only the last $2M'$ entries can be no less than $\frac{1}{2}$. Therefore there are at most $2M'$ non-zero $\mathbb{1}\left(U_{k+1}^{i,k} \geq \frac{1}{2}\right)$, thus

$$\frac{1}{i} \sum_{k=1}^i \mathbb{1}\left(U_{k+1}^{i,k} \geq \frac{1}{2}\right) \leq 2M'/i.$$

□

Meanwhile, according to Lemma D.7,

$$4 \cdot \frac{\alpha_\gamma^2}{n} \cdot \frac{1 - W_{k+1}^{i,k}}{A^{i,k}(c_{(k+1)})} \leq 8 \cdot \frac{\alpha_\gamma^2}{n} \cdot \frac{1 - U'_{k+1}}{A'(c_{(k+1)})} + U'_{k+1} \cdot \frac{M'}{i} \quad (24)$$

for all k . Then (23) can be bounded as

$$\begin{aligned} & 4 \cdot \frac{\alpha_\gamma^2}{n} \cdot \mathbb{E}[y_i^2 | S_{i+1}] + \mathbb{E}[\widehat{U}_i | S_{i+1}]^2 \\ &= 4 \cdot \frac{\alpha_\gamma^2}{n} \left(\frac{1}{i} \sum_{k=1}^i \frac{1 - W_{k+1}^{i,k}}{A^{i,k}(c_{(k+1)})} \right) + \left(\frac{1}{i} \sum_{k=1}^i W_{k+1}^{i,k} \right)^2 \\ &\leq 8 \cdot \frac{\alpha_\gamma^2}{n} \cdot \frac{1}{i} \sum_{c \in T_{i+1}} \frac{1 - U'_c}{A'(c)} + \frac{1}{i} \sum_{k=1}^i U'_{k+1} \cdot \frac{M'}{i} + (2M'/i)^2 \\ &\leq 8 \cdot \frac{\alpha_\gamma^2}{n} \cdot \frac{1}{i} \sum_{c \in T_{i+1}} \frac{1 - U'_c}{A'(c)} + \frac{M'}{i} \cdot \frac{M'}{i} + (2M'/i)^2 \\ &= 8 \cdot \frac{\alpha_\gamma^2}{n} \cdot \frac{1}{i} \sum_{c \in T_{i+1}} \frac{1 - U'_c}{A'(c)} + 5 \left(\frac{M'}{i} \right)^2. \end{aligned}$$

which completes the proof of Claim D.2. □

Claim D.4 (Compare A^* with A').

$$4 \cdot \frac{\alpha_\gamma^2}{n} \cdot \frac{1}{i} \sum_{c \in T_{i+1}} \frac{(1 - U'_c)}{A'(c)} + \left(\frac{1}{i} \sum_{c \in T_{i+1}} U'_c \right)^2 \leq 128 \sqrt{\frac{n}{i}} \cdot \frac{\alpha_\gamma^2}{n} \cdot \frac{1}{i} \sum_{c \in T_{i+1}} \frac{(1 - U_c^*)}{A^*(c)} + \left(\frac{1}{i} \sum_{c \in T_{i+1}} U_c^* \right)^2.$$

Proof. By the same reasoning as the proof of unbiased case (see Section C.3.3), it can be proved that $\frac{A^*}{32\sqrt{n/i}}, U^*$ is a feasible solution of (22). □

Combine Claim D.2 and Claim D.4,

$$\begin{aligned} & 4 \cdot \frac{\alpha_\gamma^2}{n} \cdot \mathbb{E}[y_i^2 | S_{i+1}] + \mathbb{E}[\widehat{U}_i | S_{i+1}]^2 \\ &\leq 5 \left(4 \cdot \frac{\alpha_\gamma^2}{n} \cdot \frac{1}{i} \sum_{c \in T_{i+1}} \frac{1 - U'_c}{A'(c)} + (M')^2 \right) \\ &\leq 5 \left(128 \sqrt{\frac{n}{i}} \cdot \frac{\alpha_\gamma^2}{n} \cdot \frac{1}{i} \sum_{c \in T_{i+1}} \frac{(1 - U_c^*)}{A^*(c)} + \left(\frac{1}{i} \sum_{c \in T_{i+1}} U_c^* \right)^2 \right). \end{aligned}$$

Now we compute $\mathbb{E}_{S_{i+1}} \left[4 \cdot \frac{\alpha_\gamma^2}{n} \cdot \mathbb{E}[y_i^2 | S_{i+1}] + \mathbb{E}[\widehat{U}_i | S_{i+1}]^2 \right]$ by averaging over S_{i+1} . Since S_{i+1} is a random subset, the first term of the right-hand side is just

$$\begin{aligned} 128 \sqrt{\frac{n}{i}} \cdot \frac{\alpha_\gamma^2}{n} \cdot \frac{1}{i} \left(\sum_{c \in S_{i+1}} \frac{1 - U_c^*}{A^*(c)} + \frac{1 - U_{\overline{C}}^*}{A^*(\overline{C})} \right) &= 128 \sqrt{\frac{n}{i}} \cdot \frac{\alpha_\gamma^2}{n} \left(\frac{1}{n} \sum_{j=1}^n \frac{1 - U_j^*}{A^*(c_j)} + \frac{1}{i} \cdot \frac{1 - U_{\overline{C}}^*}{A^*(\overline{C})} \right) \\ &= 32 \sqrt{\frac{n}{i}} \cdot V^* + 128 \sqrt{\frac{n}{i}} \cdot \frac{\alpha_\gamma^2}{n} \cdot \frac{1}{i} \cdot \frac{1 - U_{\overline{C}}^*}{A^*(\overline{C})}. \end{aligned}$$

Next, we upper bound the second term $\mathbb{E}_{S_{i+1}} \left[\left(\frac{1}{i} \sum_{c \in S_{i+1}} U_c^* \right)^2 \right]$. Since $S_{i+1} = \{c_1, \dots, c_i\}$ is a random subset,

$$\begin{aligned}
& \mathbb{E}_{S_{i+1}} \left[\left(\frac{1}{i} \sum_{c \in T_{i+1}} U_c^* \right)^2 \right] \\
&= \mathbb{E}_{S_{i+1}} \left[\left(\frac{1}{i} \sum_{c \in S_{i+1}} U_c^* + \frac{1}{i} \cdot U_{\bar{C}}^* \right)^2 \right] \\
&\leq \mathbb{E}_{S_{i+1}} \left[2 \left(\frac{1}{i} \sum_{c \in S_{i+1}} U_c^* \right)^2 + 2 \left(\frac{1}{i} \cdot U_{\bar{C}}^* \right)^2 \right] \\
&= \frac{2}{i^2} \cdot \mathbb{E}_{S_{i+1}} \left[\sum_{c \in S_{i+1}} (U_c^*)^2 + \sum_{c, c' \in S_{i+1}: c \neq c'} U_c^* \cdot U_{c'}^* \right] + \frac{2}{i^2} (U_{\bar{C}}^*)^2 \\
&= \frac{2}{i^2} \cdot \mathbb{E}_{S_{i+1}} \left[\sum_c \mathbb{1}(c \in S_{i+1}) (U_c^*)^2 + \sum_{c, c': c \neq c'} \mathbb{1}(c, c' \in S_{i+1}) U_c^* \cdot U_{c'}^* \right] + \frac{2}{i^2} (U_{\bar{C}}^*)^2 \\
&= \frac{2}{i^2} \cdot \frac{i}{n} \sum_{i=1}^n (U_i^*)^2 + \frac{2}{i^2} \cdot \frac{i(i-1)}{n(n-1)} \sum_{j=1}^n \sum_{k \in [n], k \neq j} U_j^* \cdot U_k^* + \frac{2}{i^2} (U_{\bar{C}}^*)^2 \\
&\leq \frac{2}{i} \cdot \frac{1}{n} \sum_{i=1}^n U_i^* + 2 \cdot \frac{(i-1)n}{i(n-1)} \cdot \frac{1}{n^2} \sum_{j=1}^n \sum_{k=1}^n U_j^* \cdot U_k^* + \frac{2}{i^2} (U_{\bar{C}}^*)^2 \\
&\leq \frac{2}{i} \cdot \frac{\sum_{i=1}^n U_i^*}{n} + 2 \left(\frac{\sum_{i=1}^n U_i^*}{n} \right)^2 + \frac{2}{i^2} (U_{\bar{C}}^*)^2.
\end{aligned}$$

Therefore

$$\begin{aligned}
& \mathbb{E}_{S_{i+1}} \left[4 \cdot \frac{\alpha_\gamma^2}{n} \cdot \mathbb{E}[y_i^2 | S_{i+1}] + \mathbb{E}[\widehat{U}_i | S_{i+1}]^2 \right] \\
&\leq 5 \cdot \mathbb{E}_{S_{i+1}} \left[128 \sqrt{\frac{n}{i}} \cdot \frac{\alpha_\gamma^2}{n} \cdot \frac{1}{i} \sum_{c \in T_{i+1}} \frac{(1-U_c^*)}{A^*(c)} + \left(\frac{1}{i} \sum_{c \in T_{i+1}} U_c^* \right)^2 \right] \\
&\leq 5 \left(32 \sqrt{\frac{n}{i}} \cdot V^* + 128 \sqrt{\frac{n}{i}} \cdot \frac{\alpha_\gamma^2}{n} \cdot \frac{1}{i} \cdot \frac{1-U_{\bar{C}}^*}{A^*(\bar{C})} + \frac{2}{i} \cdot \frac{\sum_{i=1}^n U_i^*}{n} + 2 \left(\frac{\sum_{i=1}^n U_i^*}{n} \right)^2 + \frac{2}{i^2} (U_{\bar{C}}^*)^2 \right) \\
&= 10 \left(\frac{\sum_{i=1}^n U_i^*}{n} \right)^2 + 160 \sqrt{\frac{n}{i}} \cdot V^* + \frac{10}{i} \cdot \frac{\sum_{i=1}^n U_i^*}{n} + \frac{640}{i\sqrt{i}} \cdot \frac{\alpha_\gamma^2}{\sqrt{n}} \cdot \frac{1-U_{\bar{C}}^*}{A^*(\bar{C})} + \frac{10}{i^2} (U_{\bar{C}}^*)^2.
\end{aligned}$$

Define $D_1 = 10 \left(\frac{\sum_{i=1}^n U_i^*}{n} \right)^2$, $D_2 = 160 \cdot V^*$, $D_3 = \frac{10 \sum_{i=1}^n U_i^*}{n}$, $D_4 = 640 \cdot \frac{\alpha_\gamma^2}{\sqrt{n}} \cdot \frac{1-U_{\bar{C}}^*}{A^*(\bar{C})}$, $D_5 = 10 \left(U_{\bar{C}}^* \right)^2$, then the above expectation becomes $D_1 + \frac{\sqrt{n} \cdot D_2}{\sqrt{i}} + \frac{D_3}{i} + \frac{D_4}{i\sqrt{i}} + \frac{D_5}{i^2}$. Finally, by Lemma D.9, we get

$$\begin{aligned}
& 4 \cdot \frac{\alpha_\gamma^2}{n} \cdot \mathbb{E}[\widehat{\sigma}]^2 + \left(\mathbb{E}[\widehat{U}] / n \right)^2 \\
&\leq \frac{1}{n} \sum_{i=1}^n \mathbb{E}_{S_{i+1}} \left[4 \cdot \frac{\alpha_\gamma^2}{n} \cdot \mathbb{E}[y_i^2 | S_{i+1}] + \mathbb{E}[\widehat{U}_i | S_{i+1}]^2 \right] \\
&\leq \frac{1}{n} \sum_{i=1}^n \left(D_1 + \frac{\sqrt{n} \cdot D_2}{\sqrt{i}} + \frac{D_3}{i} + \frac{D_4}{i\sqrt{i}} + \frac{D_5}{i^2} \right) \\
&\leq D_1 + 2D_2 + \frac{1 + \ln n}{n} \cdot D_3 + \frac{2}{n} (D_4 + D_5)
\end{aligned}$$

since $\sum_{i=1}^n \frac{1}{\sqrt{i}} \leq 2\sqrt{n}$ and $\sum_{i=1}^n \frac{1}{i} \leq 1 + \ln n$ and $\sum_{i=1}^n \frac{1}{\sqrt{i \cdot i}} \leq 2$. We can further replace $D_1 + \frac{1+\ln n}{n} \cdot D_3$ with a sum-of-square term $20 \left(\frac{\sum_{i=1}^n U_i^*}{n} \right)^2 + 10 \left(\frac{1+\ln n}{n} \right)^2$, which will make it easier to compare the quantity with the benchmark (details in the last paragraph of the proof).

Claim D.5.

$$D_1 + \frac{1 + \ln n}{n} \cdot D_3 \leq 20 \left(\frac{\sum_{i=1}^n U_i^*}{n} \right)^2 + 10 \left(\frac{1 + \ln n}{n} \right)^2.$$

Proof. By defintioin,

$$D_1 + \frac{1 + \ln n}{n} \cdot D_3 = 10 \left(\frac{\sum_{i=1}^n U_i^*}{n} \right)^2 + \frac{1 + \ln n}{n} \cdot \frac{10 \sum_{i=1}^n U_i^*}{n}$$

We prove the inequality by cases. First, if $\sum_{i=1}^n U_i^* \leq 1 + \ln n$ then the second term on the right-hand side $\frac{1+\ln n}{n} \cdot \frac{10 \sum_{i=1}^n U_i^*}{n} \leq 10 \frac{(1+\ln n)^2}{n^2}$. If $\sum_{i=1}^n U_i^* \geq 1 + \ln n$ then $\frac{1+\ln n}{n} \cdot \frac{10 \sum_{i=1}^n U_i^*}{n} \leq 10 \left(\frac{\sum_{i=1}^n U_i^*}{n} \right)^2$. In any cases, we get

$$D_1 + \frac{1 + \ln n}{n} \cdot D_3 \leq 20 \left(\frac{\sum_{i=1}^n U_i^*}{n} \right)^2 + 10 \left(\frac{1 + \ln n}{n} \right)^2.$$

□

Finally by plugging in all the numbers,

$$\begin{aligned} & 4 \cdot \frac{\alpha_\gamma^2}{n} \cdot \mathbb{E}[\hat{\sigma}]^2 + \left(\mathbb{E}[\hat{U}]/n \right)^2 \\ & \leq 20 \left(\frac{\sum_{i=1}^n U_i^*}{n} \right)^2 + 10 \left(\frac{1 + \ln n}{n} \right)^2 + 2D_2 + \frac{2}{n}(D_4 + D_5) \\ & \leq 20 \left(\frac{\sum_{i=1}^n U_i^*}{n} \right)^2 + 10 \left(\frac{1 + \ln n}{n} \right)^2 + 320 \cdot V^* + \frac{2}{n} \left(640 \cdot \frac{\alpha_\gamma^2}{\sqrt{n}} \cdot \frac{1 - U_C^*}{A^*(\bar{C})} + 10 (U_C^*)^2 \right) \\ & = 320 \cdot V^* + 20 \left(\frac{\sum_{i=1}^n U_i^*}{n} \right)^2 + 10 \left(\frac{1 + \ln n}{n} \right)^2 + \frac{2}{n} \left(640 \cdot \frac{\alpha_\gamma^2}{\sqrt{n}} \cdot \frac{1 - U_C^*}{A^*(\bar{C})} + 10 (U_C^*)^2 \right). \end{aligned}$$

This completes the proof of the Lemma D.10. □

Finally we compare the worst-case expected length of our confidence interval with the benchmark L^* . Our output confidence interval $\left[\sum_{i=1}^n y_i/n - \frac{\alpha_\gamma}{\sqrt{n}} \cdot \hat{\sigma}, \sum_{i=1}^n y_i/n + \frac{\hat{U}}{n} + \frac{\alpha_\gamma}{\sqrt{n}} \cdot \hat{\sigma} \right]$ has expected length $L = \mathbb{E}\left[2 \cdot \frac{\alpha_\gamma}{\sqrt{n}} \cdot \hat{\sigma} + \frac{\hat{U}}{n}\right]$. Then for any underlying cost-data distribution \mathcal{D} , L^2 can be bounded as follows using Lemma D.10 and the inequalities $(a + b)^2 \leq 2(a^2 + b^2)$, $a^2 + b^2 \leq (a + b)^2$ for all $a, b \geq 0$.

$$\begin{aligned} L^2 & = \left(2 \cdot \frac{\alpha_\gamma}{\sqrt{n}} \cdot \mathbb{E}[\hat{\sigma}] + \frac{\mathbb{E}[\hat{U}]}{n} \right)^2 \\ & \leq 2 \left(4 \cdot \frac{\alpha_\gamma^2}{n} \cdot \mathbb{E}[\hat{\sigma}]^2 + \left(\frac{\mathbb{E}[\hat{U}]}{n} \right)^2 \right) \\ & \leq 2 \left(320 \cdot V^* + 20 \left(\frac{\sum_{i=1}^n U_i^*}{n} \right)^2 + 10 \left(\frac{1 + \ln n}{n} \right)^2 + \frac{2}{n} \left(640 \cdot \frac{\alpha_\gamma^2}{\sqrt{n}} \cdot \frac{1 - U_C^*}{A^*(\bar{C})} + 10 (U_C^*)^2 \right) \right) \\ & = 640 \cdot V^* + 40 \left(\frac{\sum_{i=1}^n U_i^*}{n} \right)^2 + 20 \left(\frac{1 + \ln n}{n} \right)^2 + \frac{2560}{n} \cdot \frac{\alpha_\gamma^2}{\sqrt{n}} \cdot \frac{1 - U_C^*}{A^*(\bar{C})} + \frac{40}{n} (U_C^*)^2 \\ & \leq 640 \left(\sqrt{V^*} + \frac{\sum_{i=1}^n U_i^*}{n} \right)^2 + 20 \left(\frac{1 + \ln n}{n} \right)^2 + \frac{2560}{n} \cdot \frac{\alpha_\gamma^2}{\sqrt{n}} \cdot \frac{1 - U_C^*}{A^*(\bar{C})} + \frac{40}{n} (U_C^*)^2 \\ & = 640 \cdot \left(\frac{n+1}{n} \cdot L^* \right)^2 + 20 \left(\frac{1 + \ln n}{n} \right)^2 + \frac{2560}{n} \cdot \frac{\alpha_\gamma^2}{\sqrt{n}} \cdot \frac{1 - U_C^*}{A^*(\bar{C})} + \frac{40}{n} (U_C^*)^2. \end{aligned}$$

Since $\sqrt{\sum_j a_j} \leq \sum_j \sqrt{a_j}$,

$$L \leq 8\sqrt{10} \cdot \frac{n+1}{n} \cdot L^* + \frac{2\sqrt{5} \cdot (1 + \ln n)}{n} + \frac{16\sqrt{10}}{\sqrt{n}} \cdot \frac{\alpha_\gamma}{n^{1/4}} \cdot \sqrt{\frac{1 - U_C^*}{A^*(C)}} + \frac{2\sqrt{10}}{\sqrt{n}}.$$

This completes the proof of Theorem 5.2.