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Achieving Budget-Balance with Vickrey-Based Payment Schemes in Combinatorial Exchanges

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Abstract

Generalized Vickrey mechanisms have received wide attention in the combinatorial auction design literature because they are efficient and strategy-proof. However, it is well known that it is impossible for an exchange, with multiple buyers and sellers and voluntary participation, to be efficient and budget-balanced, even relaxing dominant strategy requirements. Except for special cases, a market-maker in an efficient exchange must make more payments than it collects. Taking a constructive approach, we clear exchanges to maximize reported surplus, and explore the efficiency effects of different budget-balanced payment rules. The payment rules are formulated to minimize the distance to Vickrey payments, under different metrics. Different rules lead to different levels of truth-revelation, and therefore efficiency. Experimental and theoretical analysis suggest a simple *Threshold scheme*, which gives surplus to agents with payments further than a certain threshold value from their Vickrey payments, has good properties. The scheme exploits agent uncertainty about bids from other agents to reduce manipulation opportunities.

A shorter version of this paper was published in the Proc. 17th International Conf. on Artificial Intelligence, 2001. [25].

1 Introduction

The participants in an exchange, or agents, can submit both *bids*, *i.e.* requests to buy items for no more than a bid price, and *asks*, *i.e.* requests to sell items for at least an ask price. Exchanges allow multiple buyers to trade with multiple sellers, with aggregation across bids and asks as necessary to clear the market. An exchange might also allow agents to express complements and substitutes through combinatorial bids and asks, together with logical conditions across bundles of different items. Combinatorial bids allow the agents to express synergies and substitutabilities between items. For example, an agent might want to buy bundle S_1 *and* sell bundle S_2, or perhaps sell S_3 *or* S_4 but not both, with no subsets of bundles acceptable for buys and no supersets acceptable for sells. Following the literature on combinatorial auctions [26, 10] we call this a combinatorial exchange.

Applications of combinatorial exchanges have been suggested to the optimization of excess steel inventory procurement [15], supply chain coordination [34], and to bandwidth exchanges [12]. The FCC has recently expressed interest in exploring two-sided market mechanisms for future wireless spectrum allocations, to enable a reallocation of licenses across both incumbents and new entrants as new wireless spectrum is introduced. One-sided combinatorial auctions are currently used for procurement and logistics problems [18], and we expect applications of combinatorial exchange technology within markets such as the Covisint B2B electronic marketplace.[1] Combinatorial exchanges are essential generalizations of one-sided combinatorial auction mechanisms.

The two core problems in an exchange are *winner-determination* (clearing), i.e. determining what is traded and by which agents; and *pricing*, i.e. determining the net payment to (or from) each agent when the exchange clears. The rules used to clear and price trades within an exchange impact the allocative-efficiency of the exchange, that is the value of trades across all agents. As an example, a mechanism that promotes truth-revelation from agents, with agents bidding their true costs and true values for bundles, elicits enough information to implement the efficient allocation.

Another key consideration in an exchange is one of *timing*. Typically, bid submission, clearing, pricing, and feedback to bidders are iterated in an exchange; with the exchange either clearing *continuously* whenever trade is possible or *periodically*. It is reasonable to expect that periodic clearing will boost the efficiency of a combinatorial exchange, allowing more opportunities for aggregations across multiple bids and asks.

In a non-combinatorial continuous double auction (CDA), such as the NYSE, bids and asks must be for a single type of item, and bids and asks are matched continuously. In the CDA the efficient trade can be priced in equilibrium with non-discriminatory item prices (everyone pays the same for the same item). The pricing problem in a combinatorial exchange is considerably more difficult in the following sense; even if we allow non-linear and

[1] www.covisint.com

discriminatory prices (not seller anonymous *or* buyer anonymous) there are some efficient trades that cannot be priced [5].

Useful economic properties of a mechanism that clears and prices an exchange include:

- **Budget-balance** (BB). The total payments received by the exchange from agents should be at least the total payments made by the exchange to agents.
- **Individual-rationality** (IR). No agent should pay more than its net increase in value for the items it trades.
- **Allocative-efficiency** (EFF). Trade should be executed to maximize the total increase in value over all agents.

In a robust implementation, these properties would be implemented in a *dominant strategy* equilibrium, such that every agent has a dominant bidding strategy whatever the preferences and strategies of the other agents. A weaker, but still useful, solution would implement these properties in a *Bayesian-Nash equilibrium*, such that every agent follows a bidding strategy that maximizes its expected utility given the bidding strategy of every other agent and distributional information about agent preferences. In addition to providing robustness to incorrect assumptions about the rationality and preferences of other agents, a dominant strategy implementation is useful computationally because agents can avoid game-theoretic reasoning about other agents [32].

Unfortunately, the well-known impossibility result of Myerson & Satterthwaite [22] demonstrates that *no* exchange can be efficient, budget-balanced, and individual-rational. This impossibility result applies even with quasi-linear utility functions[2] and Bayesian-Nash implementation. By the *revelation principle* [13, 21], this result holds for both direct, in which agents directly report their values for different trades, and indirect mechanisms, in which agents report value information indirectly, for example via responses to market prices. In addition, this result holds with or without *incentive-compatibility*, which states that the equilibrium strategy is for agents to reveal *truthful* information about their preferences. A perfect mechanism for the combinatorial exchange problem (and even the standard exchange problem) is impossible, even putting aside computational considerations.

Recognizing that combinatorial exchanges have great importance as intermediaries in electronic commerce, we take a constructive approach. We implement (BB) and (IR) as hard constraints, and select a clearing rule and payments that aims to maximize efficiency. We are interested in environments in which the market-maker must make a profit to operate the exchange, and in which participation is voluntary, so any reasonable mechanism must be (BB) and (IR). Although there will remain residual benefits for non-truthful bidding in

[2] A quasi-linear utility function assumes that an agent's utility for trade λ is a function $u_i(\lambda, p) = v_i(\lambda) - p$, where $v_i(\lambda)$ is its value for the trade, and p is the price it must pay.

any (BB) and (IR) exchange, our approach is to make strategic behavior difficult for agents with incomplete information about the preferences and strategies of other agents.

This is a paper about mechanism design, *not* about the computational complexity of clearing a combinatorial exchange. The clearing problem is NP-hard by reduction from the maximum weighted set packing problem, and approximations will be required in truly large instances, in turn further affecting efficiency and truth-revelation properties. Special-cases of the winner-determination problem, for example in which agents can receive fractional assignments and with additional restrictions on allowable types of aggregation, can be solved in polynomial time [15]. In addition, observations made about the tractability of certain special-cases of the winner-determination problem in combinatorial auctions [26, 10, 19] carry over to exchanges. In one of the few published computational studies on exchanges, Sandholm et al. [29] report that CPLEX 7.0 solves hard instances of combinatorial exchanges with multi-unit bids in less than 100 seconds for up to 10 items and 300 bids, but suggest that combinatorial exchanges may be harder to clear in practice than combinatorial auctions.

The mechanisms that we propose apply to a wide variety of combinatorial exchange settings, including exchanges with multiple units of the same items, divisible and indivisible bids, *ex ante* side constraints on feasible trades, different levels of aggregation, and with and without free-disposal. The main restriction is that we take allocative efficiency as our main objective, instead of some other objective such as trade volume.

We limit our attention to *direct mechanisms*, in which agents' bids are interpreted as claims about their values for different trades. It is useful to consider two different types of mechanisms.

(a) *Incentive-compatible mechanisms.* Design a budget-balanced and individual-rational combinatorial exchange in which truthful bidding is in equilibrium.

(b) *Non-incentive-compatible mechanisms.* Design a budget-balanced and individual-rational combinatorial exchange in which truthful bidding is not an equilibrium strategy for agents.

At first glance, mechanisms of kind (b) might seem redundant given the revelation principle, which states that any properties $\mathcal{P}$ that can be achieved with a non-incentive-compatible and perhaps indirect mechanism $\mathcal{M}'$ can be achieved with an incentive-compatible direct-revelation mechanism, $\mathcal{M}$ (see Figure 1). But the revelation principle makes a number of unreasonable computational assumptions. In particular, the revelation principle assumes that agents in the non-incentive-compatible mechanism, $\mathcal{M}'$, are able to *compute* equilibrium strategies. A constructive application of the revelation principle to mechanism design makes an implicit *worst-case* assumption about the strategic abilities agents, that they are able to exploit all opportunities for manipulation that exist in a mechanism.[3] By implement-

[3]This objection to the revelation principle is subtly different from the more standard computational

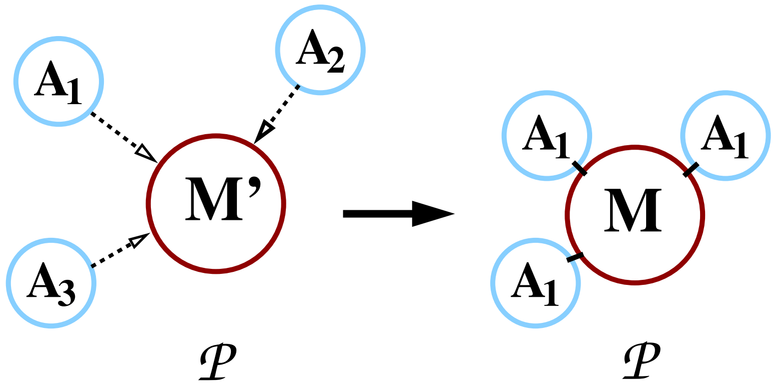


Figure 1: The revelation principle. Any properties $\mathcal{P}$ that can be implemented in Bayesian-Nash equilibrium in an indirect mechanism $\mathcal{M}'$ can be implemented in an incentive-compatible Bayesian-Nash equilibrium in a direct-revelation mechanism $\mathcal{M}$. The dashed arrows between agents $\mathcal{A}_i$ and mechanism $\mathcal{M}'$ indicate indirect revelation while the solid lines between agents and mechanism $\mathcal{M}$ indicate direct revelation.

ing an IC mechanism in which the equilibrium is *computed* for agents by the mechanism, which at that stage has good information about agent preferences, we are assuming that agents are able to compute and play an equilibrium strategy. If the actual strategies selected by agents in mechanism $\mathcal{M}'$ are off-equilibrium, then perhaps we can design $\mathcal{M}'$ such that the off-equilibrium strategy generates a better outcome (from a system-wide perspective) than is possible in an equilibrium strategy.

There is a second more pragmatic reason to prefer mechanisms of kind (b) over those of kind (a). The impossibility theorem implies that (BB) and (IR) mechanisms of kind (a), that are incentive-compatible, must deliberately implement inefficient allocations even though the agents reveal truthful information, and even though– incentive issues aside –an efficient allocation could then be implemented. Earlier strategyproof mechanisms for single-item exchanges (or double auctions) explicitly compute an inefficient outcome based on agent bids and asks [22, 20, 4]. We believe that it is often desirable to clear an exchange to maximize reported surplus, to avoid *ex post* claims from participants (that might not care about, or understand, the more subtle equilibrium arguments) that an efficient trade was forfeited.[4]

objection, see for example Ledyard et al. [17] and Parkes [24]. It is more usual to object to the use of the revelation principle as a constructive tool to focus the design of mechanisms on direct revelation instead of indirect revelation, because the theoretical indirect-to-direct transformation ignores *costs* on agents of complete information revelation (valuation and communication costs) and the computational cost on the mechanism of simulating the system and computing equilibrium outcomes.

[4]Similar issues have been debated by the FCC in the context of the optimality of allocations computed in combinatorial auctions. The FCC is concerned about legal action if a revenue-enhancing allocation was missed by the winner-determination algorithms.

In comparison, (BB) and (IR) mechanisms of kind (b) *can* clear the exchange to maximize the *reported* values (or *surplus*) of agents. These mechanisms are not incentive-compatible and agents will not necessarily reveal truthful information in equilibrium. Clearing the exchange to maximize reported value does not necessarily implement the efficient allocation and does not violate any equilibrium assumptions.

In this work we propose mechanisms of kind (b), and take the argument to its extreme. We *only* consider mechanisms that clear the exchange to maximize reported value. The mechanism design problem then reduces to designing a *payment scheme* that is budget-balanced and individual-rational, such that the incentives across agents are designed to minimize the benefits to agents of non-truthful strategic bids and boost truth-revelation. Boosting truth-revelation boosts allocative efficiency because of the clearing rule. That said, complete truth-revelation is sufficient but not necessary to implement an efficient allocation, instead information must just be truthful enough. We design payment schemes that seek to leverage the limited information available to agents about the preferences and strategies of other agents, making the holes for manipulation that must remain difficult for agents to find and exploit.

In particular, we propose a number of budget-balanced variations on the Vickrey-Clarke-Groves (VCG) [33, 7, 14] mechanism for a combinatorial exchange. The VCG mechanism is a *strategy-proof* direct-revelation mechanism, such that it is a dominant strategy for an agent to bid its true values for different trades whatever the bids and preferences of other agents. The VCG mechanism is also (IR) and (EFF), but *not* budget-balanced. Our thesis is that VCG payments continues to provide good guidance about how to implement a budget-balanced exchange. We interpret Vickrey payments as an assignment of *discounts* to agents after the exchange clears, such that a bidder makes a payment less than his bid price and a seller receives a payment greater than her ask price. Budget balance is achieved so long as the market maker distributes no more than the available surplus when the exchange clears. The pricing problem is formulated as an optimization problem, to compute discounts that minimize the distance to Vickrey discounts. We derive payment schemes that correspond to optimal solutions to this problem under a number of different distance functions. The choice of payment scheme changes the incentives within the exchange for misrepresentation of agent values, and in turn influences the efficiency of the exchange.

Theoretical and experimental analysis compares the utility to an agent for misstating its value in bids and asks in each payment scheme across a suite of problem instances. The results, both theoretical and experimental, make quite a compelling argument for a simple *Threshold* payment scheme, that provides discounts to agents with payments more than a threshold distance than their Vickrey payments, and computes the threshold to provide budget-balance.

We find it useful to discuss payment rules in terms of the residual "degree of manipulation freedom" available to an agent, which is a measure of the best-case gain from manipulation to agent in a particular mechanism. The VCG mechanism removes all *ex post* opportunities, essentially performing the misrepresentation that an agent should have done given the actual bids and asks of other agents, but at no cost to allocative efficiency. The Threshold rule allocates discounts to remove large *ex post* benefits for deviation from truthful bidding, and leaves only small *ex post* potential benefits. Leaving only small benefits makes deviations from truth-revelation less desirable to an agent with imperfect information about the other agents in a system, and helps to drive agent strategies towards truth-revelation.

We believe that the work presented here is a promising first step towards designing "second-best" mechanisms for the combinatorial exchange problem. That said, much work remains to be done to provide a full characterization of the equilibrium strategies under each rule, and to perform simulations with a richer set of agent strategies.

1.1 Example

Let us introduce an example problem, that we will return to later in the paper. Suppose agents 1, 2, 3, 4. Agents 1 and 2 want to sell A and B respectively, with values $v_1(A) = \$10$ and $v_2(B) = \$5$. Agents 3 and 4 want to buy the bundle AB, with values $v_3(AB) = \$51$ and $v_4(AB) = \$40$. The efficient allocation is for agents 1 and 2 to trade with agent 3, for a net increase in value of \$36.

The mechanism design problem is: given bid and ask prices for A, B and AB from the agents, what trades should take place and what payments should be made and received?

Here are a number of reasonable budget-balanced payment rules:

– charge agent 3 its bid price, pay agents 1 and 2 their ask price.

– divide the surplus equally, charging agent 3 a price of \$39, and providing payments of \$22 and \$17 to agents 1 and 2 respectively.

– provide the surplus to the buyer, charging agent 3 a price of \$15 and providing payments equal to bid price to agents 1 and 2.

– divide the surplus across the sellers in proportion to their ask prices; i.e. agent 1 receives $\$10 + (10/15)\$36 = \$34$ and agent 2 receives $\$5 + (5/15)\$36 = \$17$.

In this paper we develop a family of payment rules, and analyze their effect on the allocative efficiency of an exchange. Payment rules have an indirect effect on allocative efficiency via their effect on agent strategies.

2 Combinatorial Exchanges

The essential element that defines a combinatorial exchange is that the bids and asks of agents are expressive enough to describe values over *bundles* of items. This expressibility allows an agent to represent both complements and substitutes across items.

A simple combinatorial bidding language might allow agents to submit bids and asks explicitly for bundles of items. A *bid*, $B = (S, p_{\text{bid}})$, associates a *bid price* $p_{\text{bid}} \geq 0$ with a bundle of items $S \subseteq \mathcal{G}$, where $\mathcal{G}$ is the set of all items in the exchange. This is the most an agent will pay for bundle S. An *ask*, $A = (S, p_{\text{ask}})$, associates an *ask price* $p_{\text{ask}} \geq 0$ with a bundle of items S. This is the minimum payment an agent will accept for bundle S. Such a bidding language can also be readily extended to allow agents to bid for *multiple units* of items. Logical predicates can be introduced to connect these simple bidding elements. One example is "additive-or" connectors, that allow an agent to state that any number of a set of bids and/or asks can be selected simultaneously by the market-maker. Another example is "exclusive-or" connectors, that allow an agent to state that at most one of a set of bids and/or asks can be selected.

Stepping back from the specifics of the bidding language, any language will induce values over different *trades*. Let $\mathcal{I}$ denote the set of agents.

Definition 1 (Trade). *A trade* $\lambda_i = (\lambda_i(1), \ldots, \lambda_i(|\mathcal{G}|))$*, where* $\lambda_i(j) \in \mathbb{Z}$ *is an integer, defines a transfer of* $\lambda_i(j)$ *units of item* $j \in \mathcal{G}$ *to agent* i *if* $\lambda_i(j) > 0$*, and a transfer of* $\lambda_i(j)$ *units of item* j *from agent* i *if* $\lambda_i(j) < 0$*.*

Let Λ denote the set of all possible trades. Each agent can be engaged in both buying and selling items, so a trade allows the exchange of items in both directions, depending on the sign of $\lambda_i(j)$. We also allow trades of multiple units of the same good, i.e. $\lambda_i(j)$ is not restricted to just {-1, 0, 1} but can be any integer value. Although we refer to trades over integer values of items, all the mechanisms that are proposed in this paper extend immediately to problems with fractional allocations of items.

Bids and asks induce a *reported* value, $\hat{v}_i(\lambda_i) \in \mathbb{R}$, for every trade $\lambda_i \in \Lambda$.

Definition 2 (Valuation function). *A valuation function* $\hat{v}_i(\lambda_i)$ *denotes an agent's net value for the trade.*

Bids indicate positive value for buying a bundle of items, while asks indicate negative value for selling a bundle of items. This explicit representation of the value of a trade does not allow an agent to specify a preference for the *identity* of the buyer or seller with which it executes its trade. However this is without loss of generality, because we can introduce the identity of an agent into the good space with a "dummy item" appended to the bids and asks of that agent, essentially converting attributes into sets of products. Additional side constraints (e.g. a limit on the number of winners, on the volume of trade, etc.) can also be implemented directly within the clearing rules of the exchange.

Returning to our simple bidding language, suppose there are goods $\mathcal{G} = \{A, B, C\}$ and consider bid $(AB, 10)$ and ask $(C, 5)$ from agent 1. These bids induce valuation function, $\hat{v}_1([1, 1, 0]) = 10, \hat{v}_1([0, 0, -1]) = -5, \hat{v}_1([1, 1, -1]) = 5$. The values for other trades are constructed to be consistent with value $-\infty$ for selling anything other than item C, a zero value for buying $S \subset \{AB\}$, and no additional value for buying more than bundle AB.

2.1 Clearing

As discussed in the introduction, exchanges can clear either periodically, with intervals between periods measured for example in time or in bid volume, or continuously, with any feasible trade cleared immediately. Periodic clearing probably increases allocative-efficiency in combinatorial exchanges, because it provides an opportunity to aggregate across a larger number of bids and asks. In addition, the payment schemes that we present in this paper are most applicable to periodic clearing, because that presents the most choice for surplus distribution. The NYSE is a continuous double auction, in which traders submit bids and asks for immediate execution. The Arizona Stock Exchange (AZX) is a periodic, or *call* market, in which bids and asks are accumulated and cleared at periodic, pre-specified intervals [31]. Call markets are also used to open sessions in continuous markets (e.g. the Bourse de Paris), and used for less active securities and bonds.

Bids may be *open* or *closed*. This refers to the information that is passed on to the bidders by the market maker. In an open bid market all agents may observe all the submitted bids (probably without the identity of the bidder), they may also learn what are the winning bids and how much the winners pay/ are paid. In a closed (or sealed) bid market the agents have no knowledge of the other agents' bids and learn only whether their own bids won or not and their own payment. There are many possibilities between these two extremes; for instance, submitted bids are not revealed to other agents, everyone is notified of his/her trade in the current provisional outcome and non-winners are provided with a minimum bid increment (or ask decrement) to enter the provisional trade if other bids remain unchanged. Open bids, and general market transparency, may encourage better coordination between agents, but also allow more manipulation in thin markets.

The NASDAQ is an example of an exchange with open bids, although trading information is not evenly distributed across participants; access to information about bids and asks waiting to clear (in the "limit book") is distributed among market brokers and held until the trades are cleared. A new plan, the Next Nasdaq, calls for the implementation of a centralized database of bids and asks that is visible to all market participants [31]. In typical procurement examples one would expect closed bids because the information contained in bids may well present useful information to competitors [9].

Following the discussion in the introduction, in this paper we are interested in mechanisms in which the overall trade that is implemented given a set of bids and asks is that which maximizes reported value (or surplus). We define this as the winner determination problem.

Definition 3 (winner determination problem). *The winner determination problem in a combinatorial exchange is to compute the overall trade that maximizes revenue given a set of bids and asks.*

Formally, let $\lambda = (\lambda_1, \ldots, \lambda_{|\mathcal{I}|}) \in \times_{i \in \mathcal{I}} \Lambda$ denote a complete trade across all agents. Given reported valuation functions, $\hat{v}_i(\lambda_i)$ for trades λ_i, a general mathematical programming formulation for the winner determination problem is:

$$\lambda^* = \arg \max_{\lambda \in \times_i \Lambda} \sum_{i \in \mathcal{I}} \hat{v}_i(\lambda_i)$$
$$\text{s.t.} \quad \mathit{feasible}(\lambda)$$

where *feasible*(λ) are a set of constraints to define whether or not the trade is feasible.

A feasible trade must respect supply and demand, for example without allocating the same item to more than one agent. The exact interpretation of *feasible* depends on, amongst other things:

- *Aggregation*: the role of the market-maker in disassembling and reassembling bundles of items.
- *Divisibility*: the ability to allocate fractions of items, and the ability to satisfy a fraction of agents' bids and asks. When an agent wants its bid or nothing, then its bid is called *indivisible*.
- *Ex-ante constraints*: for example, on the volume of trade, assignment constraints, market concentration, etc.
- *Free-disposal*: the ability to match a superset of a bundle of items with an ask, and/or a subset of a bundle of items with a bid.

In combination with divisibility, free-disposal allows a bundle of items to be matched with the maximal fraction of an agent's bid that remains a subset of the bundle of items. The payment rules introduced in this paper are valid in all of these cases.

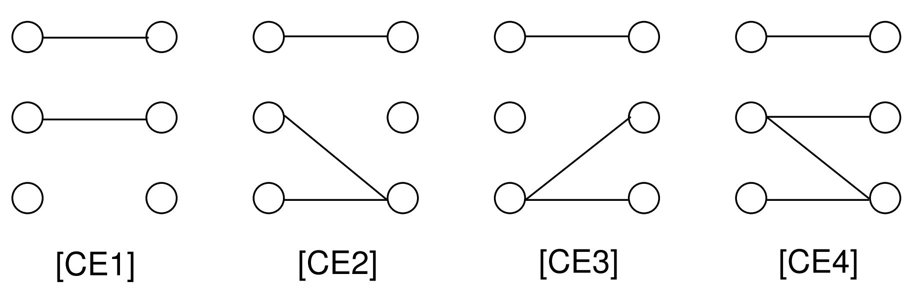


Figure 2: Models of Aggregation: [CE1] no aggregation or disaggregation; [CE2] buy-side aggregation; [CE3] sell-side aggregation; [CE4] complete aggregation.

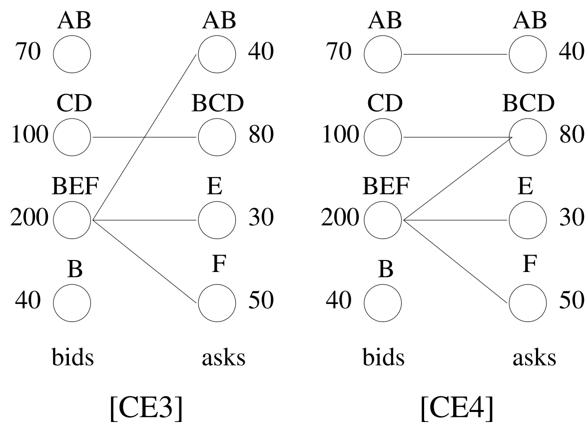


Figure 3: Example 1. Clearing in a combinatorial exchange: [CE3]: sell-side aggregation, $V^* = 100$; [CE4]: complete aggregation, $V^* = 170$.

Possible models of aggregation are illustrated in Figure 2. The nodes represent the collection of bids and asks submitted by a single agent. Moving from [CE1] to [CE4] the auctioneer has a more active role, and the maximum possible efficiency of the outcome increases. For example, in [CE1] all trades are bilateral and there is no auctioneer intervention, while in [CE4] there is complete aggregation. The appropriate level of aggregation will depend on the *physical attributes* of the good; e.g. pieces of steel can be cut but not very easily joined (buy-side aggregation [CE2]), conversely computer memory chips can be combined but not split (sell-side aggregation [CE3]). Note that aggregation *does not* imply that the exchange must take physical possession of goods, trades can still be executed directly between agents.

As an example, Figure 3 illustrates the trades executed for a simple problem, under aggregation models [CE3] and [CE4]. The exchange receives 4 bids to buy bundles, and 4 bids and 4 asks, and there are 6 items, $\{A, \ldots, F\}$ and no logical constraints across bids and asks. We assume free disposal, such that excess items in an ask can be stripped away at no cost, either by the seller, the buyer, or the exchange. As expected, the surplus in [CE4] is greater than in [CE3], \$170 compared with \$100. For comparison, the solution for [CE2] is to match ask $(BCD, \$80)$ with bids $(CD, \$100)$ and $(B, \$40)$, for surplus \$90. The optimal bilateral match is ask $(AB, \$40)$ with bid $(AB, \$70)$ and ask $(BCD, \$80)$ with bid $(CD, \$100)$, for a surplus of \$50.

The computational complexity of clearing an exchange depends on the interpretation of feasibility, and in addition on whether bids and asks are single units of items or for multiple-units. The base case, with one item, divisible bids, aggregation possible on both sides of the market, and no constraints is polynomial time solvable by equating supply and demand; the problem remains polynomial time solvable with multiple items. Kalagnanam et al. [15] show that introducing assignment constraints does not make the problem computationally more difficult, the winner-determination problem can be formulated as a network-flow problem. In general, as soon as bids are indivisible the problem becomes NP hard by reduction from maximum weighted set packing. However, if there is no aggregation then the problem can be solved as a maximum weight matching problem in polynomial time irrespective of divisibility. If aggregation is possible on any one side of the exchange and there is at least one indivisible bid and/or ask the problem becomes NP-hard. The complexity analysis for variations that allow piecewise-linear demand and supply curves are discussed in Eso et al. [11] and Sandholm & Suri [28] for simplified one-sided markets. In addition, observations made about the tractability of certain special-cases of the winner-determination problem in combinatorial auctions [26, 10, 19] carry over to exchanges.

In practice, despite the NP-hardness of winner-determination in one-sided combinatorial auctions, off-the-shelf general purpose Integer Programming solvers, such as OSL or CPLEX, have been successful in solving quite large winner-determination problem in one-sided combinatorial auctions (with 1000's of items and bids), even for what are believed to be hard distributions [1, 28]. Moreover, recent studies on approximation algorithms (using an approximate linear-programming algorithm together with greedy hill-climbing heuristics) demonstrate solution times comparable with the time to send problem instances over a T1 line [37], and with average-case efficiency within 4% of optimal. In one of the few published computational studies on solving the winner-determination problem in combinatorial exchanges, in this case with full aggregation and indivisible bids, Sandholm et al. [29] conjecture that two-sided problems might be more difficult to solve in practice than one-sided problems.

We cautiously put the computational complexity of winner-determination to one side in this paper, and assume optimal solutions in describing or pricing mechanisms. Although this is reasonable for tractable special cases and small problem instances, approximations will be necessary in large problems. In general one must be careful when introducing approximate solutions within mechanisms, because the presence of an approximation algorithm can break the incentive compatibility of a mechanism [23]. This secondary effect on allocative-efficiency is less of a concern in this paper because we do not try to design fully incentive-compatible mechanisms.

2.2 Pricing

The pricing problem in an exchange is to determine the payments made by agents to the exchange and made by the exchange to agents after the exchange clears. In exchanges such as the NYSE the price is simply one of the bid or ask prices, essentially allocating the entire surplus from the trade to one of the agents. This simple rule does not extend to a combinatorial exchange with anything beyond bilateral matching (aggregation [CE1]) because multiple parties can be involved in a single trade.

Definition 4 (payment). *The payment $p(i)$ to agent i is the price paid by the agent to the exchange if $p(i) > 0$, and the price paid by the exchange to the agent if $p(i) < 0$.*

As is very common in the literature on mechanism design and auction theory, we assume that agents have a quasi-linear utility function.

Definition 5 (quasi-linear utility). *A utility function $u_i(\lambda_i, p(i))$ is quasi-linear if $u_i(\lambda_i, p(i)) = v_i(\lambda_i) - p(i)$, where $p(i)$ is the agent's payment and $v_i(\lambda_i)$ is its value for trade $\lambda_i \in \Lambda$.*

Implicit in this assumption is that agents are *risk neutral*, with the same expected utility for a certain payment of \$10 and a trade with an expected value of \$10 and zero price.

In equilibrium, payments should satisfy individual-rationality (IR), such that no agent ever has negative utility from participation in the exchange, and budget-balance (BB) or individual-rationality for the exchange itself, such that the exchange never makes more payments to the agents than the total payments made by agents to the exchange. Technically, this is *ex post* IR and *ex post* BB, in future work it would be interesting to explore whether relaxing IR to *interim*, or relaxing BB to *ex ante* would improve the allocative efficiency of mechanisms for combinatorial exchanges.[5]

For a class of equilibria in which agents only *overstate* values for items they are selling and *understate* values for items they are buying, we can ensure *ex post* individual-rationality with the constraint:

$$\hat{v}_i(\lambda_i^*) - p(i) \geq 0, \quad \forall i \in \mathcal{I} \tag{IR}$$

where $\hat{v}_i(\lambda_i^*)$ is the reported value to agent i of the trade λ_i^* that is implemented when the exchange clears.[6] As a special case, truthful strategies are individual-rational with payments that satisfy this constraint.

For *ex post* weak budget-balance we require that the total payments to the exchange are non-negative:

$$\sum_{i \in \mathcal{I}} p(i) \geq 0 \tag{BB}$$

[5]For example, an extension to the Groves mechanism, the *dAGVA* (or "expected Groves") mechanism, due to Arrow [2] and d'Aspremont and Gerard-Varet [8], demonstrates that it is possible to achieve efficiency and *ex post* budget-balance with *interim* individual-rationality, such that agents will voluntarily participate if they must decide to participate even before they know their own preferences. This might be a reasonable approximation in settings in which participants have costly valuation problems and costly participation costs.

[6]In particular, if $\hat{v}_i(\lambda_i) > 0$ then $p(i)$ can be positive, but less than $\hat{v}_i(\lambda_i)$, and the exchange must request a payment of no more than an agent's reported value for the implemented trade. Similarly, if $\hat{v}_i(\lambda_i) < 0$ then $p(i)$ must be negative, and the exchange must pay the agent at least its reported loss in value from the trade.

It is useful to express an agent's payment, $p(i)$, as a *discount* from its reported value:

Definition 6 (Discount). *The discount Δ_i to agent i is the reported surplus to agent i at trade λ_i given payment $p(i)$, i.e. $\Delta_i = \hat{v}_i(\lambda_i) - p(i)$.*

Given this definition, equivalent definitions of (IR) and (BB) are:

$$\Delta_i \geq 0, \quad \forall i \in \mathcal{I} \tag{IR'}$$

and

$$\sum_{i \in \mathcal{I}} \hat{v}_i(\lambda_i) \geq \sum_{i \in \mathcal{I}} \Delta_i \tag{BB'}$$

In Section 4 we formulate the pricing problem as a discount-allocation problem, in which the objective is to minimize the distance between discounts and discounts under the VCG scheme for some objective function, subject to (IR') and (BB').

3 Mechanism Design for Exchanges

This section presents a number of preliminaries. First, we define the VCG for a combinatorial exchange and briefly discuss some conditions on agent preferences and models of aggregation underwhich (BB) does and does not hold. Then, we review some mechanisms that have been proposed to address this impossibility for the standard non-combinatorial exchange, or double auction. This helps to provide some context for the proposed combinatorial exchange mechanism.

3.1 The VCG Mechanism

The Vickrey-Clarke-Groves (VCG) [33, 7, 14] mechanism for an exchange computes the revenue-maximizing trade, λ^*, with all bids and asks, and also the revenue-maximizing trade $(\lambda_{-i})^*$, with each agent $i \in \mathcal{I}$ taken out of the exchange in turn. Let V^* denote the revenue from trade λ^*, and $(V_{-i})^*$ denote the revenue from the maximal trade without agent i. Trade λ^* is implemented, and each agent makes payment $p_{\text{vick},i} = \hat{v}_i(\lambda_i^*) - (V^* - (V_{-i})^*)$. Negative payments $p_{\text{vick},i} < 0$ indicate that the agent *receives* money from the exchange after it clears.

Definition 7 (Vickrey discount). *The Vickrey discount to agent i is $\Delta_{\text{vick},i} = V^* - (V_{-i})^*$.*

The Vickrey discount, which is the amount by which an agent's bid price (or ask price) is discounted, is always non-negative, providing smaller payments to the exchange for agents with reported *positive* net values for the trade, and larger payments from the exchange for agents with *negative* net values for the trade.

Proposition 1 (individual-rational). *The VCG mechanism is individual rational, such that the expected utility to rational agents from participation is also non-negative.*

Proof. It is easy to show that $V^* \geq (V_{-i})^*$ by a simple feasibility argument, so that the Vickrey discount is always non-negative. Truthful bidding is the rational strategy for an agent, so a rational agent will always pay less than its net value for the trade. □

In addition, truth-revelation is a dominant strategy equilibrium.

Proposition 2 (strategy-proof). *The VCG mechanism is strategy-proof.*

Proof. Consider the utility to agent i for reported valuation $\hat{v}_i$.

$$\begin{aligned} u_i(\hat{v}_i) &= v_i(\lambda_i^*) - p_{\text{vick},i} \\ &= v_i(\lambda_i^*) + \sum_{j \neq i} \hat{v}_j(\lambda_j^*) - (V_{-i})^* \end{aligned}$$

Ignoring the final term, which is independent of agent i's bid, agent i should announce a valuation $\hat{v}_i$ to maximize the sum of its own *actual* value for the trade $\lambda^* = (\lambda_1^*, \ldots, \lambda_I^*)$ and the *reported* value to the other agents. Given that trade λ^* is computed to maximize the total reported value of the agents, agent i should announce $\hat{v}_i = v_i$ to align the computation of the market maker in solving the clearing problem with its own interests. □

The following proposition follows as an immediate corollary of strategy-proofness, because the trade λ^* is computed to maximize reported value.

Proposition 3 (efficient). *The VCG mechanism is efficient.*

In equilibrium, with truth-revelation from agents, the utility to an agent is captured by its Vickrey payoff (which is identical to its Vickrey discount).

Definition 8 (Vickrey payoff). *The Vickrey payoff to agent i is $\pi_{\text{vick},i} = u_i(\lambda_i^*, p_{\text{vick},i}) = v_i(\lambda_i^*) - (v_i(\lambda_i^*) - \Delta_{\text{vick},i})$, or simply $\pi_{\text{vick},i} = V^* - (V_{-i})^*$.*

Figure 4 and 5 provide some intuition about the strategy-proofness of the VCG mechanism. In Figure 4 we plot the agent's utility vs. its reported value on the efficient trade, λ_i^*, and compare the utility in the VCG mechanism with a "pay-what-you-bid", or *No Discount* mechanism. Notice that the agent's utility in the Vickrey mechanism is unchanged while its reported value is high enough to implement the efficient trade as the reported surplus maximizing trade. In comparison, in the No Discount mechanism the agent can maximize its utility by bidding just the minimal value that still implements the efficient trade.

This is generalized in Figure 5, in which we plot the agent's utility for different reported valuation functions. For simplicity we map all valuation functions into a single dimension. In this general case, for reported valuation functions too far from truthful, the utility in the VCG mechanism jumps down to lower plateaus as the implemented trade changes. As before, in the No Discount mechanism the agent can maximize its utility by reporting just the right value.

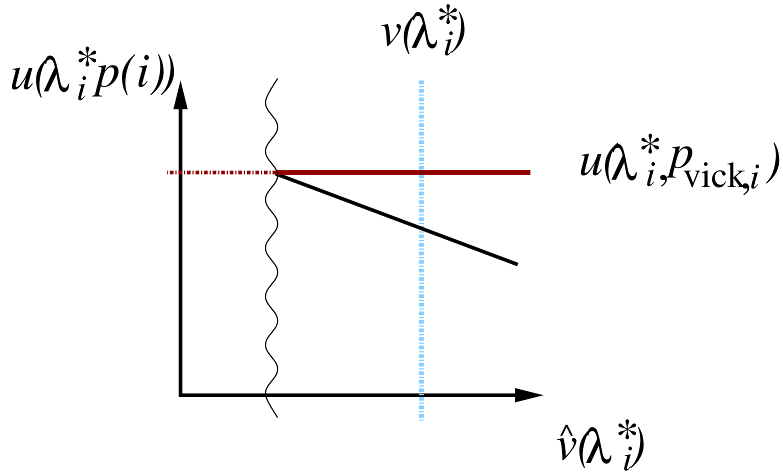


Figure 4: Agent utility against bid price, with all bids except λ_i^* truthful. To the left of the wavy line the maximal trade changes. The horizontal line is the agent's utility in the VCG mechanism, while the inclined line is an agent's utility in a No Discount mechanism.

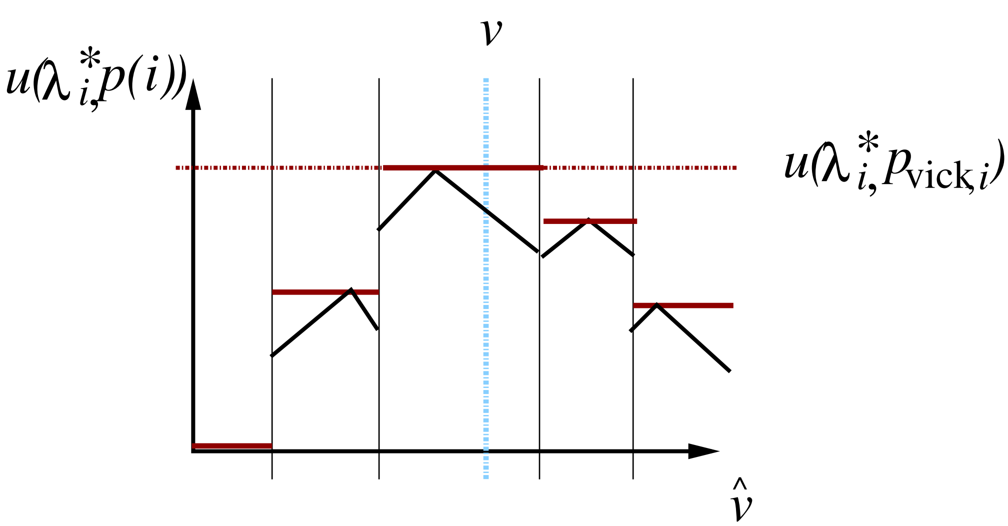


Figure 5: Agent utility against bid price, with all possible reported valuations mapped onto a single axis. The flat plateaus represent utility in the VCG mechanism; the jumps coincide with a change in the maximal trade. The inclined lines represent utility under a No Discount mechanism for different bids.

3.2 Budget Balance and Efficiency: Tension

The Myerson-Satterthwaite impossibility theorem [22] shows that the joint goals of (BB), (IR) and (EFF) are essentially impossible to achieve. The formal statement of the impossibility result is made for *ex ante* budget-balance (i.e. even allowing average-case budget-balance) and *interim* individual-rationality (i.e. even allowing average-case non-negative expected utility from participation). This negative result extends via the revelation principle to indirect and non incentive-compatible mechanisms, and includes dominant-strategy implementations and ex-post individual-rationality as special cases.

Following Krishna & Perry [16], we find it convenient to prove the impossibility through a connection with the VCG mechanism. Let us state a special case of an important property of the VCG mechanism:

Theorem 1 (Payment Maximization). *Among all mechanisms that are efficient, interim individual-rational, and Bayesian-Nash incentive-compatible, the VCG mechanism maximizes the expected payments of each agent.*

Proof. See Krishna & Perry [16]. □

It follows that there exists an ex-ante (BB), efficient, and interim individual-rational mechanism for an allocation problem *only* in the case that the VCG mechanism for the problem has non-negative expected revenue. This provides a convenient proof of the Myerson-Satterthwaite impossibility result.

Theorem 2 (Myserson-Satterthwaite impossibility). *It is impossible to achieve allocative-efficiency, ex ante budget-balance and interim individual-rationality in a Bayesian-Nash incentive-compatible mechanism, even with quasi-linear utility functions.*

Proof. [16] Consider a situation of trade with a single seller (agent 1) and a single buyer (agent 2), and one indivisible item owned by the seller. Agent 1 has value $v_1([-1]) \in [0, 1]$ for trade $\lambda_1 = [-1]$ and $v_1([0]) = 0$ for no trade. Agent 2 has value $v_2([+1]) \in [0, 1]$ for trade $\lambda_2 = [+1]$, and $v_2([0]) = 0$ for no trade. As shorthand, let $v_1([-1]) = v_1$ and $v_2([+1]) = v_2$. To compute the expected revenue to the market-maker with the VCG mechanism consider the following two cases:

(a) $v_1 + v_2 \leq 0$. In this case $\lambda_1^* = \lambda_2^* = [0]$ and $p_{\text{vick},1} + p_{\text{vick},2} = 0$.

(b) $v_1 + v_2 > 0$. In this case $\lambda_1^* = [-1]$ and $\lambda_2^* = [+1]$, and $p_{\text{vick},1} + p_{\text{vick},2} = v_1 - (V^* - (V_{-1})^*) + v_2 - (V^* - (V_{-2})^*) = v_1 - (v_1 + v_2) + 0 + v_2 - (v_1 + v_2) + 0 = -(v_1 + v_2) < 0$.

Therefore, whenever there is a non-zero probability of gains from trade then the expected payment from agents is negative, the VCG mechanism runs at a deficit, and (ex ante) budget-balance fails. The result follows from the saliency of VCG. □

The impossibility result sets up an interesting second-best design problem. Exchanges, and in particular combinatorial exchanges, have many exciting applications that require mechanisms. But, any feasible design must relax one of (BB), (IR) and (EFF). As we discussed in the introduction, we require (IR) and (BB) and accept some loss in allocative-efficiency.

3.3 Vickrey Budget-Balance: Success & Failure

In this section we characterize conditions that are sufficient for budget-balance with Vickrey payments in an exchange. In fact, we will see that budget-balance failure is quite pervasive with Vickrey payments.

Example

Consider the Vickrey payments in the problem introduced in Section 1.1. In this case, assuming that agents submit these bids in the VCG mechanism, then we have $V^* = 51 - 10 - 5 = 36$, $(V_{-1})^* = (V_{-2})^* = 0$, $(V_{-3})^* = 25$, and $(V_{-4})^* = 36$. The Vickrey payment to agent 1 is $p_{\text{vick},1} = -10 - (36 - 0) = -46$, the Vickrey payment to agent 2 is $p_{\text{vick},2} = -5 - (36 - 0) = -41$, and the Vickrey payment to agent 3's is $p_{\text{vick},3} = 51 - (36 - 25) = 40$. The exchange runs at a loss of \$47 to the market maker.

Budget-balance requires that the sum of the Vickrey discounts across all agents is no greater than the available surplus when the exchange clears.

$$V^* \geq \sum_{i \in \mathcal{I}} (V^* - (V_{-i})^*) \qquad \text{(Vickrey-BB)}$$

The total marginal surplus contribution of each agent must be no greater than the total surplus of the agents taken together. This is a special case of the decreasing marginal returns requirement introduced by Bikchandani & Ostroy [5] to characterize combinatorial allocation problems in which Vickrey payments can be supported in competitive equilibrium.

Immediately, we have the following proposition:

Proposition 4. *If $(V_{-i})^* = 0$ for any agent $i \in \mathcal{I}$ then VCG payments are (BB) if and only if we also have $(V_{-j})^* = V^*$ for every other agent $j \neq i$.*

In other words, if there is one agent that is *critical*, in the sense that there are no revealed gains-from-trade in its absence, then there must be *perfect competition* across the other agents, in the sense that their presence has *no* marginal effect on the possible gains from trade.

It is interesting to strengthen (Vickrey-BB) by enforcing the following per-agent constraint:

$$v_i(\lambda_i^*) \geq V^* - (V_{-i})^*, \quad \forall i \in \mathcal{I} \qquad \text{(Vickrey-BB')}$$

In other words, the marginal contribution of each agent to the revealed surplus is less than its own value for the trade. It is immediate that condition (Vickrey-BB') implies condition (Vickrey-BB). A necessary condition for this stronger budget-balance requirement is that:

$$v_i(\lambda_i^*) \geq 0, \quad \forall i \in \mathcal{I}$$

because $V^* - (V_{-i})^* \geq 0$. Now, suppose that Vickrey discounts are made to a subset $\mathcal{J} \subseteq \mathcal{I}$ of agents, while other agents receive no discount. Given that $v_i(\lambda_i^*) \geq 0$ for all agents, then [R1] is sufficient for budget-balance:

[R1] Agent j is a *buyer* but not a seller of items, and either free-disposal is possible by the market maker, or always possible by some agent $k \neq j$. This is sufficient because the remaining trades remain feasible at no loss in surplus, and $(V_{-j})^* \geq V^* - v(\lambda_j^*)$.

We can also apply condition [R1] recursively, within a cluster in a safe partition, and apply Vickrey discounts to all buyers in a cluster as long as there is no agent in that cluster with negative value for its trade, and as long as there is free-disposal. As an example, the VCG applied to a combinatorial allocation problem, with one seller and multiple buyers, is budget-balanced. Vickrey discounts are allocated to all bidders in the auction, but the seller receives the sum of the payments from the bidders and not her Vickrey payment. This is budget-balanced by condition [R1] whenever there is free disposal, because the seller is not assumed to have a reservation price for the items. The budget-balance with this one-sided payment rule is quite tight; as soon as the seller receives her Vickrey discount then *every* buyer must pay its bid price.

A decomposition technique is useful to characterize another budget-balanced special case. Suppose that $(\mathcal{C}_1, \ldots, \mathcal{C}_N)$ is a complete partition over agents $\mathcal{I}$. Weaker than (Vickrey-BB'), but still sufficient for budget balance is:

$$\sum_{i \in \mathcal{C}_k} v_i(\lambda_i^*) \geq \sum_{i \in \mathcal{C}_k} (V^* - (V_{-i})^*), \quad \forall k \in \{1, \ldots, N\}$$

Let us define a *safe partition*:

Definition 9 (safe partition). *A partition $(\mathcal{C}_1, \ldots, \mathcal{C}_N)$ is safe if the trades within each cluster $\mathcal{C}_k$ are feasible.*

In a safe partition, removing one or more agents within any one cluster only affects trade feasibility within that cluster, and does not have a ripple effect on the feasibility of agents elsewhere in the partition. It is budget-balanced to allocate Vickrey discounts to agents in set $\mathcal{J} \subseteq \mathcal{I}$ in the following special case:

[R2] No two agents $l, m \in \mathcal{I}, l \neq m$ are in the same cluster, i.e. only one agent in each cluster receives a Vickrey discount.

To see this, notice that $(V_{\mathcal{I} \setminus \mathcal{C}_k})^* \geq V^* - \sum_{i \in \mathcal{C}_k} v_i(\lambda_i^*)$ because the trades executed by agents outside of the cluster remain feasible. It follows that $\sum_{i \in \mathcal{C}_k} v_i(\lambda_i^*) \geq V^* - (V_{\mathcal{I} \setminus \mathcal{C}_k})^* \geq V^* - (V_{-l})^* = \sum_{i \in \mathcal{C}_k} (V^* - (V_{-i})^*)$, where $l \in \mathcal{C}_k$ is the agent in the cluster that receives its Vickrey discount.

As an example, consider a modified VCG for bilateral matching, in which one agent per match receives its Vickrey discount and the other agent receives no discount. This mechanism is budget-balanced by [R2], because the pairs of matched agents define a safe partition.

Notice that neither [R1] or [R2] help in the earlier example of a combinatorial exchange problem in which VCG payments are not budget-balanced. Case [R1] is not useful because the sellers have negative values for the trade. Case [R2] does not add anything, because it is always budget-balanced to provide the Vickrey discount to at most one agent (Proposition 4), and the only safe partition in the example is the trivial partition into clusters $(\{1, 2, 3\}, \{4\})$.

3.4 Budget-Balanced Mechanisms for Single-item Exchanges

Budget-balanced mechanisms for the standard multiple-unit single-item exchange (or double auction) provide useful context for the proposed family of VCG-based combinatorial exchange mechanisms. The clearing and payment problem in a double auction with periodic clearing is illustrated in Figure 6. Assume that bids are sorted in descending order, such that $B_1 \geq B_2 \geq \ldots \geq B_n$, while asks are sorted in ascending order, with $A_1 \leq A_2 \leq \ldots \leq A_m$. The efficient trade is to accept the first $l \geq 0$ bids and asks, where l is the maximal index for which $B_l \geq A_l$. The problem is to determine which trade is implemented, and agent payments.

In a full VCG mechanism for the double auction, the successful buyers make payment $\max(A_l, B_{l+1})$ and the successful sellers receive payment $\min(A_{l+1}, B_l)$. In general the payments are not budget-balanced, for example with $A_{l+1} < B_l$ and $B_{l+1} > A_l$ and $A_{l+1} > B_{l+1}$.

Theorem 3. *Budget-balance is achieved in a simple exchange for homogeneous items and single-item bids and asks if and only if one (or more) of the following conditions hold: (1) $p^0_{bid} = p^0_{ask}$; (2) $p^0_{bid} = p^{-1}_{bid}$; (3) $p^0_{ask} = p^{-1}_{ask}$.*

Proof. BB holds if and only if $\max(p^0_{ask}, p^{-1}_{bid}) \geq \min(p^0_{bid}, p^{-1}_{ask})$, leading to cases: (1) $p^0_{ask} \geq p^{-1}_{bid}$ and $p^0_{bid} \leq p^{-1}_{ask}$; (2) $p^0_{ask} < p^{-1}_{bid}$ and $p^0_{bid} \leq p^{-1}_{ask}$; (3) $p^0_{ask} \geq p^{-1}_{bid}$ and $p^0_{bid} > p^{-1}_{ask}$. □

In other words, either one or more of the supply or demand curves must be "smooth" at the clearing point, with the first excluded bid at approximately the same bid price as the last accepted bid, *or* the winning bid and ask prices must precisely coincide.

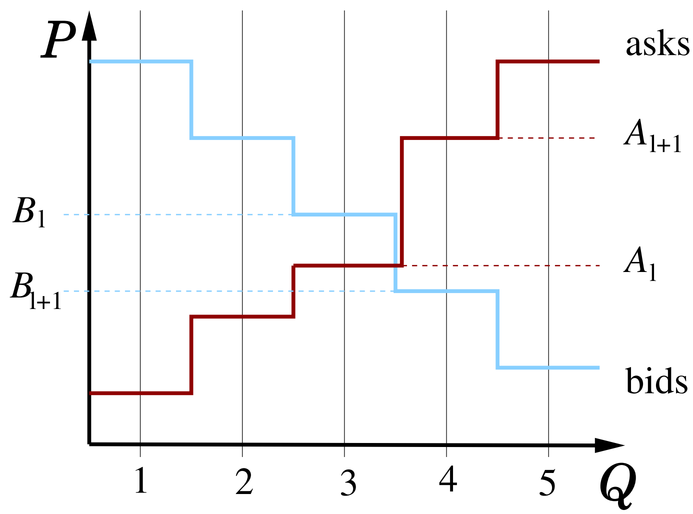


Figure 6: A standard double auction. Bid price vs. Ask price for successive quantities, $\mathcal{Q}$, of the same item. In this example it is efficient to trade $l = 3$ units, and there is a non-discriminatory clearing price $p \in [A_l, B_l]$.

Table 1 surveys double auction mechanisms known in the literature. As expected by the Myerson-Satterthwaite impossibility theorem, no mechanism is (EFF), (BB) and (IR). All mechanisms except the VCG-DA are (BB) and (IR) but not (EFF), and all except the k-DA mechanism are strategy-proof.[7] In the k-DA [35, 6, 30], parameter $k \in [0, 1]$ is chosen before the auction begins; the parameter is used to calculate a clearing price somewhere between A_l and B_l. The McAfee DA [20] computes price $p^* = (A_{l+1} + B_{l+1})/2$, and implements this price if $p^* \in [A_l, B_l]$ and trades l units, otherwise $l-1$ units are traded for price B_l to buyers and A_l to sellers. Babaioff & Nisan [3] have recently proposed the TR-DA rule, which is just the fall-back option of McAfee's DA.[8]

We earlier classified (BB) and (IR) mechanisms for exchanges into mechanisms of kind (a), that are (IC) and deliberately clear the exchange to implement an inefficient trade, and kind (b), that are not (IC) and clear the exchange to implement the revealed-surplus maximizing trade. Mechanisms TR-DA and McAfee-DA fall into kind (a), while mechanism

[7] Recently, Yoon [36] has proposed a modified version of the VCG-DA in which participants are charged a fee to enter the auction and balance the budget-loss of the VCG payments. Yoon characterizes conditions on agents' preferences under which the modified VCG-DA is (EFF), (IR) and (BB).

[8] Babaioff & Nisan also propose an α-reduction DA (not included in the table), in which a parameter $\alpha \in [0, 1]$ is selected before the auction begins. The TR-DA rule is used with probability α, and the VCG DA rule is used with probability $1 - \alpha$. Parameter α can be chosen to make the expected revenue zero (and achieve *ex ante* BB) with distributional information about agent values, to balance the expected surplus loss in the VCG-DA with expected gain in the TR-DA. The α-reduction DA is (BB) and (IR) but not (EFF), with dominant-strategy incentive-compatibility.

Name	traded	p_{buy}	p_{ask}	(EFF)	(BB)	(IR)	equil	(IC)
VCG-DA	l	$\max(A_l, B_{l+1})$	$\min(A_{l+1}, B_l)$	Yes	No	Yes	dom	yes
k-DA [6, 35]	l	$kA_l + (1-k)B_l$	$kA_l + (1-k)B_l$	No	Yes	Yes	Nash	no
TR-DA [3]	$l-1$	B_l	A_l	No	Yes	Yes	dom	yes
McAfee-DA [20]	l or $l-1$	$(A_{l+1} + B_{l+1})/2$ or B_l	$(A_{l+1} + B_{l+1})/2$ or A_l	No	Yes	Yes	dom	yes

Table 1: Double auction mechanisms. The *traded* column indicates the number of trades executed, where l is the efficient number. The *equil* column indicates whether the mechanism implements a dominant strategy or (Bayesian)-Nash equilibrium.

k-DA falls into kind (b).

4 Vickrey-Based Budget-Balanced Payment Rules

Our thesis is that the VCG provides useful information to design a budget-balanced exchange. We propose a family of VCG-based exchanges in which the exchange is cleared to implement the trade that maximizes reported value (or surplus), and we construct payments that minimize the distance to VCG payments for some metric and also satisfy (IR) and (BB) constraints. The choice of distance function is shown to have a distributional effect on the allocation of surplus and changes the incentive-compatibility properties of the exchange.

In this section we do the following:

- Formulate the pricing problem as a mathematical program, to minimize the distance to Vickrey payments with (BB) and (IR) as hard constraints.
- Introduce a number of possible distance functions and construct corresponding budget-balanced payment schemes.
- Derive some theoretical properties that hold for the rules, and present an example application of the rules to the problem instance introduced in the introduction.

In Sections 5.1 and 5.2 we present a theoretical analysis of each payment scheme for a couple of simple models, and in Section 6 we present an experimental analysis of each payment scheme within a slightly more realistic environment.

4.1 A Surplus-Allocation Model for the Pricing Problem

The pricing problem is to use the available surplus, V^*, computed at value-maximizing trade λ^*, to allocate discounts to agents that have good incentive properties while ensuring (IR) and (BB). Consider the graphical illustration in Figure 7. We must allocate the available

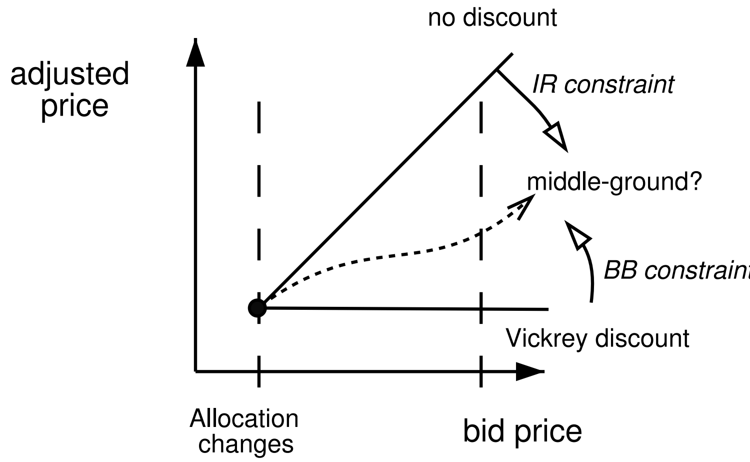


Figure 7: The discount allocation design space. With "no discount", an agent's adjusted price decreases as its bid approaches the smallest price that is still sufficient to implement the same trade. The Vickrey discount provides a flat adjusted price, that is the same for a range of bid prices.

discounts across agents that are less than the Vickrey discount in some places, to achieve (BB), and never negative to ensure (IR).

We formulate this problem as a linear program, to allocate surplus to agents to minimize distance to Vickrey discounts. Let V^* denote the available surplus when the exchange clears, before any discounts; let $I^* \subseteq \mathcal{I}$ denote the set of agents that trade. We compute discounts $\mathbf{\Delta} = (\Delta_1, \ldots, \Delta_I)$ to minimize the distance $\mathbf{L}(\mathbf{\Delta}, \mathbf{\Delta}_{\text{vick}})$ to Vickrey discounts, for a suitable distance function $\mathbf{L}$.

$$\min_{\mathbf{\Delta}} \quad \mathbf{L}(\mathbf{\Delta}, \mathbf{\Delta}_{\text{vick}}) \qquad \text{[PP]}$$

$$\text{s.t.} \quad \sum_{i \in I^*} \Delta_i \leq V^* \qquad \text{(BB')}$$

$$\Delta_i \leq \Delta_{\text{vick},i} \quad, \forall\, i \in I^* \qquad \text{(VD)}$$

$$\Delta_i \geq 0 \quad, \forall\, i \in I^* \qquad \text{(IR')}$$

Notice that the discounts are *per-agent*, not per bid or ask, and therefore apply to a wide range of bidding languages (as described in Section 2). Each agent may perform a number of buys and sells, depending on its bids and asks and the bids and asks of other agents. Constraints (BB') and (IR') over the discount allocation were first introduced in Section 2.2. (BB') gives worst-case (or ex post) budget-balance, such that the exchange never makes a net payment to agents. This constraint can be easily strengthened if it is desirable for the market-maker to take a sliver of the surplus before computing the discounts. We might also substitute an expected surplus $\overline{V}^*$ for V^* and implement average-case (or ex ante) budget-balance.

The (IR') constraints ensure that truthful bids and asks are (ex post) individual-rational for an agent, such that an agent has non-negative utility for participation whatever the bids and asks received by the exchange.[9] Constraints (VD) ensure that no agent receives more than its Vickrey discount. The constraints are not redundant for certain distance metrics, such as the $\mathbf{L}_{\text{RE}}(\cdot)$ metric.

In addition to the standard $\mathbf{L}_2(\boldsymbol{\Delta}, \boldsymbol{\Delta}_{\text{vick}}) = \sum_i (\Delta_{\text{vick},i} - \Delta_i)^2$ and $\mathbf{L}_\infty(\boldsymbol{\Delta}, \boldsymbol{\Delta}_{\text{vick}}) = \max_i (\Delta_{\text{vick},i} - \Delta_i)$ distance metrics, we also consider the following functions: (a) $\mathbf{L}_{\text{RE}}(\boldsymbol{\Delta}, \boldsymbol{\Delta}_{\text{vick}}) = \sum_i \frac{\Delta_{\text{vick},i} - \Delta_i}{\Delta_{\text{vick},i}}$, a relative error function; (b) $\mathbf{L}_\Pi(\boldsymbol{\Delta}, \boldsymbol{\Delta}_{\text{vick}}) = \prod_i \frac{\Delta_{\text{vick},i}}{\Delta_i}$, a product error function; (c) $\mathbf{L}_{\text{RE2}}(\boldsymbol{\Delta}, \boldsymbol{\Delta}_{\text{vick}}) = \sum_i \frac{(\Delta_{\text{vick},i} - \Delta_i)^2}{\Delta_{\text{vick},i}}$, a squared relative error function; and (d) $\mathbf{L}_{\text{WE}}(\boldsymbol{\Delta}, \boldsymbol{\Delta}_{\text{vick}}) = \sum_i \Delta_{\text{vick},i}(\Delta_{\text{vick},i} - \Delta_i)$, a weighted error function. Divide by zero is avoided in all distance functions by dropping agents with $\Delta_{\text{vick},i} = 0$, and simply setting $\Delta_i = 0$ for these agents. The $\mathbf{L}_1$ metric is not interesting, providing no distributional information because any complete allocation of surplus is optimal. Of course, these functions do not represent an exhaustive set of distance measures. One could also consider rules used in combination with pre-selection, in which agents are dropped from consideration using an initial rule, and perhaps stochastically.

4.2 Payment Rules

Rather than solving problem [PP] directly, we can compute an analytic expression for the family of solutions that correspond to each distance function. Each family of solutions is a parameterized *payment rule*. The payment rules are presented in Table 2. We include the *Equal* rule, which divides the available surplus equally across all agents $I^* \subseteq \mathcal{I}$ that trade, and the *No-Discount* or "pay-what-you-bid" rule. In the Appendix we derive the payment rules and the methods to compute optimal parameterizations of each rule using the methods of Lagrangian optimization.

Each payment rule is parameterized, for example the Threshold rule, $\Delta_i^*(C_t) = \max(0, \Delta_{\text{vick},i} - C_t)$, depends on the selection of Threshold parameter C_t. The final column in Table 2 summarizes the subproblem that must be solved to determine the optimal parameterization for each rule. For example, the optimal Threshold parameter, C_t^*, is selected as the smallest C_t for which the discount allocation remains budget-balanced. The optimal parameter for any particular rule is typically not the optimal parameter for another rule.

Let us suppose that the agents are indexed such that their Vickrey discounts are decreasing, with $\Delta_{\text{vick},I} \leq \Delta_{\text{vick},I-1} \leq \ldots \leq \Delta_{\text{vick},2} \leq \Delta_{\text{vick},1}$, and introduce additional points $\Delta_{\text{vick},0} = \infty$ and $\Delta_{\text{vick},I+1} = 0$, and index the interval $[\Delta_{\text{vick},k+1}, \Delta_{\text{vick},k}]$ by k. We can

[9] We choose to design for the truth-revealing equilibrium, so this is an appropriate representation of (IR). In fact, this is slightly pessimistic if agents are expected to shave their actual bids in practice.

Distance Function	Name	Definition	Parameter Selection
$\mathbf{L}_2$, $\mathbf{L}_\infty$	Threshold	$\max(0, \Delta_{\text{vick},i} - C_t^*)$	$\min C_t$ s.t. (BB')
$\mathbf{L}_{\text{RE}}$	Small	$\Delta_{\text{vick},i}$, if $\Delta_{\text{vick},i} < C_s^*$ 0 otherwise	$\max C_s$ s.t. (BB')
$\mathbf{L}_{\text{RE2}}$	Fractional	$\mu^* \Delta_{\text{vick},i}$	$\mu^* = V^* / \sum_i \Delta_{\text{vick},i}$
$\mathbf{L}_{\text{WE}}$	Large	$\Delta_{\text{vick},i}$, if $\Delta_{\text{vick},i} > C_l^*$ 0 otherwise	$\min C_l$ s.t. (BB')
$\mathbf{L}_\Pi$	Reverse	$\min(\Delta_{\text{vick},i}, C_r^*)$	$\max C_r$ s.t. (BB')
-	No-Discount	0	-
-	Equal	$V^* / \lvert I^* \rvert$	-

Table 2: Distance Functions, Payment Rules, and optimal parameter selection methods. Constraint (BB') states that $\sum_i \Delta_i^* \leq V^*$, and $|I^*|$ (used in the Equal rule) is the number of agents that participate in the trade.

provide an intuitive description of the payments implemented in each rule, given optimal parameterizations C_l^*, C_s^*, μ^*, C_l^*, and C_r^*:

- **Threshold**. If parameter C_t^* falls into interval k, then agents $i = 1, \ldots, k$ will receive discount $\Delta_i^*(C_t^*) = \Delta_{\text{vick},i} - C_t^*$, and agents $i = k+1, \ldots, I$ will not receive any discounts.

- **Small**. If parameter C_s^* falls into interval k, then agents $i = 1, \ldots, k$ will not receive any discount while agents $i = k+1, \ldots, I$ will receive their Vickrey discounts.

- **Fractional**. Every agent i receives discount $\mu^* \Delta_{\text{vick},i}$, where $\mu^* = V^* / \sum_i \Delta_{\text{vick},i}$; i.e. agents receive a fraction of the surplus equal to their proportional share of total discount under the VCG mechanism.

- **Large**. If parameter C_l^* falls into interval k then agents $i = k+1, \ldots, I$ will not receive any discounts while agents $i = 1, \ldots, k$ will receive their Vickrey discounts.

- **Reverse**. If parameter C_r^* falls into interval k then agents $i = 1, \ldots, k$ will receive their Vickrey discount while agents $i + 1, \ldots, I$ receive a discount in the amount of C.

Let us sketch the construction of the Threshold rule from distance metric $\mathbf{L}_2$. Introducing Lagrange multiplier, $\lambda \geq 0$, we have $\min \sum_i (\Delta_{\text{vick},i} - \Delta_i)^2 + \lambda(\sum_i \Delta_i - V^*)$, s.t. $0 \leq \Delta_i \leq \Delta_{\text{vick},i}$. Now, computing first derivatives with respect to Δ_i and setting to zero, we have $-2(\Delta_{\text{vick},i} - \Delta_i^*) + \lambda = 0$ for all i, where Δ_i^* is the optimal allocation of discount to agent i.[10] Solving, this equalizes the difference between Vickrey discounts and actual

[10] First-order conditions are necessary and sufficient for optimality in this problem because the Hessian is positive definite.

discounts across all agents with $\Delta_i^* > 0$, i.e. $\Delta_{\text{vick},1} - \Delta_1^* = \Delta_{\text{vick},2} - \Delta_2^* = \ldots$. Parameter $C_t \geq 0$ denotes this difference, and gives budget balance for $C_t^* = (\sum_{i=1}^{K} \Delta_{\text{vick},i} - V^*)/K$, where index K is such that $\Delta_{\text{vick},K+1} \leq C_t^* \leq \Delta_{\text{vick},K}$. Every agent with Vickrey discount greater than parameter C^* receives a discount $\Delta_{\text{vick},i} - C_t^*$, while the other agents receive no discount.

Example

In Table 3 we compare the payments made with each payment scheme in the simple problem that we presented in the introduction. The payments are all (IR) and (BB) (except for the Vickrey mechanism), but each rule allocates a different discount to each agent.

Rule	Vick	Equal	Frac	Thresh	Reverse	Large		Small	
Agent 1	-46	-22	-25.6	-28	-22.5	-46	or -10	-35	or -10
Agent 2	-41	-17	-20.6	-23	-17.5	-5	-41	-5	-30
Agent 3	40	39	46.2	51	40	51	51	40	40

Table 3: Payments with Different Rules in the Simple Problem.

One thing to notice is that the Large and Threshold rules allocate no discount to the successful buyer, agent 3, and divide all surplus across the sellers. In comparison, the Equal rule actually provides more than the Vickrey discount to agent 3, a problem which is fixed with Reverse. In contrast with Large and Threshold, the Small and Reverse rules allocate the maximal discount to the one buyer and divide the remaining surplus across the two sellers.

It is also interesting that in this simple example neither the Large or Small schemes provide useful guidance about how to distribute the discount across the two sellers because this depends on how the tie is broken.

4.3 Graphical Representations

In Figure 8 we present a graphical illustration of the effect of each payment rule on the adjusted price for different levels of bid prices. Let λ_i^* denote the trade to agent i when it bids truthfully, i.e. with $\hat{v}_i = v_i$, and suppose that the agent's reported value, $v = v_i(\lambda_i^*)$ for the trade is positive. Consider the effect of bidding $b_i \neq v_i(\lambda_i^*)$ on trade λ_i^*, but leaving the reported values for other trades truthful. Let $x = p_{\text{vick},i} = v - \Delta_{\text{vick},i}$, denote a critical value, such that: $V^* - (v - x) = (V_{-i})^*$, and any bid price $b_i < x$ for trade λ_i^* must change the surplus maximizing trade. We assume that $(\lambda_{-i})^*$ is the maximal trade for bid prices less than x.

To keep things simple, assume that trade λ^* continues to solve the winner-determination problem for all bids down to x:

$$\lambda^* = \arg \max_{\lambda \in \times_i \Lambda} \left[\hat{v}_i(\lambda_i) + \sum_{j \neq i} v_j(\lambda_j) \right]$$
$$\text{s.t.} \quad \textit{feasible}(\lambda), \quad \text{and } b \geq x$$

where $\hat{v}_i$ is truthful except for $\hat{v}_i(\lambda_i^*) = b$. We also choose to ignore for the moment any effect that a change in bid price can have on the optimal parameterization for a particular payment rule, or the available surplus.

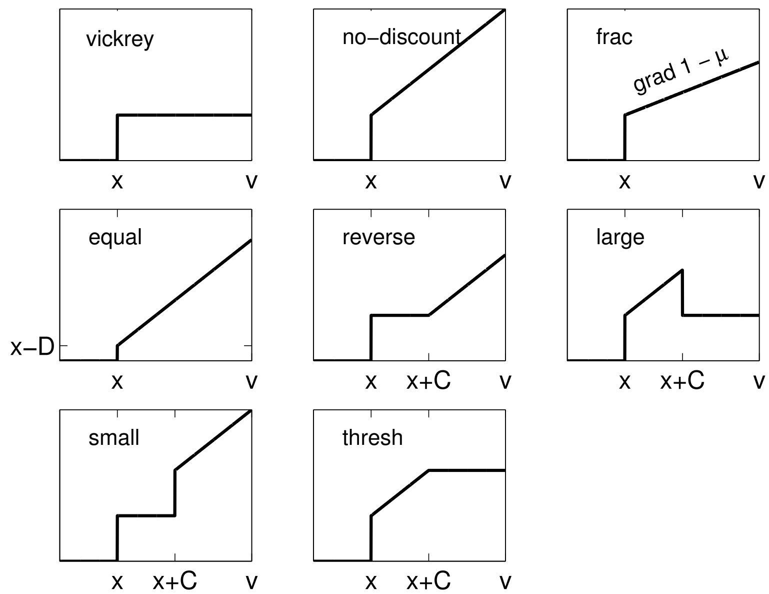


Figure 8: Bid price b_i on trade λ_i^* against adjusted bid price $p(b)$ in each payment scheme. Agent i has value $v > 0$ for λ_i^*, and value x is the agent's Vickrey payment, also the smallest bid price at which trade λ_i^* is executed. The parameters, C, indicate the optimal selections for each rule, and need not be the same across rules; similarly parameter μ is the optimal selection for the Fractional rule. Parameter D in Equal is a share $D = V^* / |I^*|$ of the available surplus.

Figure 8 illustrates the adjusted payment to agent i under different rules, as its bid price b changes. For values $b < x$, the payment is zero under all rules because the agent is not in the trade. For $b \geq x$, the Vickrey discount is $\Delta_{\text{vick},i}(b) = V^*(b) - (V_{-i})^*$, where $V^*(b)$ is the surplus from the maximal trade with bid b. Simplifying, $\Delta_{\text{vick},i}(b) = V^* - (v - b) - (V^* - (v - x)) = b - x$. Clearly, the adjusted payment, $p(b) = b - \Delta_{\text{vick},i} = x$, under the Vickrey rule is independent of the bid price. Briefly, here are the derivations of the plots for each rule:

(a) **Fractional**. $p(b) = b - \mu^* \Delta_{\text{vick},i} = b - \mu^*(b - x) = b(1 - \mu^*) + \mu^* x$.

(b) **Equal**. $p(b) = b - V^* / |I^*| = b - D$

(c) **Reverse**. $p(b) = b - \min(b - x, C_r^*)$, and $p(b) = b - C_r^*$ if $b > C_r^* + x$, or $p(b) = x$ otherwise.

(d) **Large**. $p(b) = b - (b - x) = x$, if $b > C_l^* + x$, or $p(b) = b$, otherwise.

(e) **Small**. $p(b) = b$, if $b > C_s^* + x$, or $p(b) = x$ otherwise.

(f) **Threshold**. $p(b) = b - \max(0, b - x - C_t^*)$, and $p(b) = x + C$ if $b > x + C_t^*$, or $p(b) = b$ otherwise.

Looking at the plots of payment vs. bid price, one could conjecture from the strategy-proofness of the VCG mechanism that flat regions provide good incentives for truth-revelation. The situation is a little more involved because it is possible than an agent's own bids and change both the height and the position of the flat region in the non-VCG payment schemes, via a weak-coupling through the budget-balance constraint. In the next section we present the results of some simple analysis, first identifying conditions underwhich the flat regions do not shift, and then incorporating a simple model of how the flat regions change based on an agent's own bids. In Section 7 we attempt to tie this all back to the plots in Figure 8, and draw some conclusions about good positioning of these flat regions.

5 Theoretical Analysis

In this section we first present we characterize sufficient conditions on the state of an exchange for truth revelation to be the rational strategy for an agent, under different payment rules, where the state of an exchange is defined by the bids and asks from other agents. In addition, in Section 5.2 we construct a simple model for the interaction between an agent's bids and asks and the parameterizations of a payment rule (that determine, for example, the position and height of flat regions), and compare the truthfulness of agent strategies at the equilibrium of this simple model, again for different payment rules.

None of the analysis is for a full equilibrium model, in which agents play best-response strategies to the best-response strategies of other agents. Extending these first analytic steps to a full equilibrium analysis is left as important future work.

5.1 Static Best-Response Analysis

First, consider that the bids and asks of other agents are fixed, and define via the payment rules in a mechanism a utility-maximization problem for an agent. Our analysis will assume that an agent has perfect information about the bids and asks of other agents, and therefore be worst-case from the perspective of a mechanism designer.

Fix the bids and asks from other agents, and consider a single buyer with a positive value for the trade, $\lambda^*(v_i)$, that is computed to clear the exchange when it reports its true value. Using λ_i^* to denote $\lambda_i^*(v_i)$, in other words we have $v_i(\lambda_i^*) > 0$. Let $\hat{v}_i$ denote some non-truthful bidding strategy of agent i.

It is useful to introduce the concept of the *residual degree of manipulation freedom*, RDMF(i), to agent i in a mechanism.

Definition 10 (residual degree of manipulation freedom). *Fix the bids and asks from other agents. Then the residual degree of manipulation freedom,* RDMF(i)*, for agent i, is the maximal amount that the agent can increase its utility with some bid $\hat{v}_i \neq v_i$, in comparison to its utility from the outcome when it bids truthfully.*

This is a measure of the ex post (i.e. given agents bids and asks, and the allocated discounts) opportunity for manipulation in a payment scheme. Clearly, RDMF(i) = 0 in the VCG mechanism, and RDMF(i) = $\Delta_{\text{vick},i}$ with the No Discount rule.

Proposition 5. *The difference, $\Delta_{\text{vick},i} - \Delta_i$, is an upper-bound on the RDMF(i) in a VCG-based exchange mechanism.*

Proof. The maximal payoff available to agent i is its Vickrey payoff. To see this, notice that the trade computed by all VCG-based mechanisms is the same trade as is computed in the VCG mechanism, and that for any outcome the agent cannot receive a greater discount than its Vickrey discount, from the (VD) constraint. □

In fact, this is a tight upper-bound on RDMF(i) in a VCG-based exchange mechanism.

Proposition 6. *No VCG-based exchange mechanism can achieve a RDMF(i) better than $\Delta_{\text{vick},i} - \Delta_i$.*

Proof. Construct bid $\hat{v}_i(\lambda_i^*) = v_i(\lambda_i^*) - (V^* - (V_{-i})^*)$, and $\hat{v}_i(\lambda) = 0$ on all $\lambda \neq \lambda_i^*$; where λ_i^* is the trade to agent i in the VCG mechanism, given the bids and asks from the other agents. The VCG-based mechanism will implement the VCG outcome for agent i. □

Putting this all together, consider VCG-based exchange mechanism, $\mathcal{M}$, and let $u_{\text{vick},i}$ denote agent i's utility with truth-revelation in the VCG mechanism, and let $u_i(\mathcal{M})$ denote the agent i's utility with truth-revelation in $\mathcal{M}$. The RDMF is computed as the maximal gain in utility available to an agent:

$$\begin{aligned}\text{RDMF}(i) &= u_{\text{vick},i} - u_i(\mathcal{M}) \\ &= [v_i(\lambda_i^*) - (v_i(\lambda_i^*) - \Delta_{\text{vick},i})] - [v_i(\lambda_i^*) - (v_i(\lambda_i^*) - \Delta_i)] \\ &= \Delta_{\text{vick},i} - \Delta_i\end{aligned}$$

where $0 \leq \Delta_i \leq \Delta_{\text{vick},i}$ is the discount allocated to agent i in the payment rule $\mathcal{M}$.

This provides an interesting reinterpretation of the objective functions with which each payment rule is derived. In particular, for the Threshold rule:

Proposition 7. *The Threshold rule minimizes the maximal RDMF amongst all Vickrey-based (IR) and (BB) payment schemes.*

Immediately, the upper-bound on the payoff to an agent in a VCG-based mechanism provided by its Vickrey payoff provides a number of special cases in which truth-revelation is optimal for an agent, given the bids and asks of other agents. In each case the agent receives its full Vickrey discount with a truthful bid. Of course, these propositions say nothing about agent equilibrium strategies, beyond providing a necessary condition for a truth-revealing Nash equilibrium.

Proposition 8. *Fix bids and asks of other agents. Truthful bidding is the utility-maximizing strategy for an agent in Large whenever $\Delta_{\text{vick},i} \geq C_l^*$.*

Proposition 9. *Fix bids and asks of other agents. Truthful bidding is the utility-maximizing strategy for an agent in Small whenever $\Delta_{\text{vick},i} \leq C_s^*$.*

Proposition 10. *Fix bids and asks of other agents. Truthful bidding is the utility-maximizing strategy for an agent in Reverse whenever $\Delta_{\text{vick},i} \leq C_r^*$.*

We would like to make similar but slightly weaker claims for in other cases, for example that *any* agent that receives a non-zero Vickrey discount in the Threshold mechanism cannot reduce its payment without bidding closer to its critical value than the Threshold parameter. Referring back to Figure 8, the subtle problem with making strong statements about the incentive-compatibility of the Threshold rule for agents that are in the "flat region", and receiving discounts, is that as an agent changes its bids the optimal threshold parameter, C_t^*, can change to maintain budget-balance.[11]

We can identify some special cases on the state of the exchange in which the effect of the bids and asks of other agents, coupled with the payment rule in an exchange mechanism, are to provide a useful decoupling between an agent's bid price and its payment. Let the *maximal trade without agent* j, denoted λ^{-j}, refer to the trade that is implemented without any bids or asks from agent j. Suppose that agent j bids truthfully for all trades except λ_i^*, and let $\hat{v}_i(\lambda_i^*) = v_i(\lambda_i^*) - \theta$ for some $\theta \in \mathbb{R}$ denote the agent's adjusted bid for trade λ_i^*. Consider the following special cases:

[C1] Trade, λ_i^*, to agent i is implemented in the maximal trade without agent j, for all $j \neq i$, for both truthful and untruthful bids, $\hat{v}_i$, from agent i; i.e. $\lambda_i^* = \lambda_i^{-j} = \lambda_i^{-j}(\hat{v}_i)$, for all $j \neq i$.

[C2] If the agent submits a higher bid for λ_i^*, i.e. with $\theta < 0$, then it increases its reported value for other trades by at most $|\theta|$.

[11] An agent can reduce its adjusted payment, without bidding low enough to achieve $\Delta_{\text{vick},i} < C_t^*$, by lowering the threshold C_t^*. Recall that $C_t^* = (\sum_{i=1}^{K} \Delta_{\text{vick},i} - V^*)/K$, where index K is such that $\Delta_{\text{vick},K+1} \leq C_t^* \leq \Delta_{\text{vick},K}$.

[C3] There exists a higher bid, $\hat{v}_i(\lambda_i^*)$, for some $\theta < 0$, such that λ_i^* is in the maximal trade without agent j, for all agents $j \neq i$.

It is convenient to denote the new values for surplus, Vickrey discounts, and optimal rule parameters at bids $\hat{v}_i \neq v_i$, as $V^*(\theta)$, $\Delta_{\text{vick},i}(\theta)$, and $C_t^*(\theta)$, etc.

We have the following positive results:

Proposition 11. *If [C1] holds, then an agent that receives a non-zero discount in Threshold cannot benefit from a lower bid unless $\theta > \Delta_{\text{vick},i} - C_t^*$, or unless the implemented trade changes.*

Proof. We show that the optimal Threshold does not change, and that the agent's Vickrey discount remains to the right of the Threshold. The new Vickrey discount $\Delta_{\text{vick},i}(\theta) = \Delta_{\text{vick},i} - \theta$, and $\Delta_{\text{vick},i}(\theta) > C_t^*$ because $\theta \leq \Delta_{\text{vick},i} - C_t^*$. Agent i's discount is θ less than with truth-revelation, exactly compensating for the θ drop in surplus to distribute. In addition, $C_t^*(\theta) = C_t^*$, because the Vickrey discounts are unchanged for all agents $j \neq i$: we have $\Delta_{\text{vick},j}(\theta) = (V^* - \theta) - ((V_{-j})^* - \theta) = \Delta_{\text{vick},j}$ by [C1]. □

Proposition 12. *If [C2] holds, than an agent in Threshold cannot benefit from submitting a higher bid unless the implemented allocation changes.*

Proof. Clearly, $\Delta_{\text{vick},i}(\theta) = \Delta_{\text{vick},i} + \theta$, and there is a neutral effect on the numerator of C_t^* due to the change in surplus and discount to agent i. The discounts to other agents $j \neq i$ can only increase, because $V^*(\theta) = V^* + \theta$, and $(V_{-j})^*(\theta) \leq (V_{-j})^* + \theta$ by [C2]. Therefore, C_t^* can increase but not decrease, leaving an agent with $\Delta_{\text{vick},i} > C_t^*$ at best indifferent. In addition, C_t^* cannot increase by more than θ, and bring agent i into the Threshold region, because at best $\Delta_{\text{vick},j}(\theta) - \Delta_{\text{vick},j} \leq \theta$ for every agent j with $\Delta_{\text{vick},j} \geq C_t^*$. □

Proposition 13. *If [C3] holds, an agent in Large that receives no discount with truthful bids can always achieve its Vickrey payoff, with some suitably high reported value on λ_i^*.*

Proof. First, $\Delta_{\text{vick},i}(\theta) = \Delta_{\text{vick},i} + \theta$. Then, for some $\theta < M$, for some finite $M > 0$, trade λ_i^* is implemented by agent i in the maximal trade without agent $j \neq i$, for all j by assumption [C3]. Writing $\theta = M + \theta_1$, then $V^*(\theta) = V^* + M + \theta_1$ and $(V_{-j})^*(\theta) \geq (V_{-j})^* + \theta_1$, so that $\Delta_{\text{vick},j}(\theta) \leq V^* + M - (V_{-j})^* = \Delta_{\text{vick},j} + M$. Therefore, there is some finite θ for which $\Delta_{\text{vick},i}(\theta) > \Delta_{\text{vick},j}(\theta)$ for all $j \neq i$, and agent i has the largest Vickrey discount. In this state, if any agent receives its Vickrey discount with Large it will be agent i, and there is always enough surplus to provide the Vickrey discount to any one agent (Prop. 4). □

5.2 A Partial (Aggregated) Equilibrium Analysis

In this section we consider the best-response of agents with respect to a simple model of each exchange, that aggregates the effect of the bids and asks of other agents on the outcome of the exchange, but does model the effect that an agent's own bids and asks can have on the parameters in a particular payment scheme, via the requirements for budget-balance. The analysis proves very useful, providing strong intuition about the effects that drive incentives for truthful bidding in each scheme.

To keep things simple, we focus again on a single agent, i, and suppose that agent i misrepresents its value on some trade, λ_i^*. We consider two interesting cases. First, we consider an agent that trades some items when it submits a truthful bid. In this case the trade on which the agent misrepresents its value is the trade implemented with truth-revelation. Second, we consider an agent that does not trade with truthful bids. In this case the trade on which an agent misrepresents its value will be the trade, if any, that is implemented by the exchange from some bid. Let $v_i = v_i(\lambda_i^*)$ denote the agent's true value, and $b_i = \hat{v}_i(\lambda_i^*) = v_i - \theta$ denote the agent's bid, with level of manipulation $\theta \in \mathbb{R}$.

The effect of the bids and asks of other agents is captured in the aggregate with two parameters: the critical value, and the available surplus to distribute to agents as discounts.

Definition 11 (critical value). *The critical value, x_{crit}, is the smallest bid price that agent i can bid for λ_i^* and still implement the same trade.*

In the first case, when trade λ_i^* is implemented with truthful bids, then the critical value is defined with expression $V^* - v_i + x_{\text{crit}} = (V_{-i})^*$, from which $x_{\text{crit}} = v_i - (V^* - (V_{-i})^*) = p_{\text{vick},i}$, and $x_{\text{crit}} < v_i$. In the second case, when trade λ_i^* is not implemented with truthful bids, the critical value is defined with expression $(V_{+1})^* - v_i + x_{\text{crit}} = V^*$, from which $x_{\text{crit}} = v_i - ((V_{+i})^* - V^*)$, where $(V_{+i})^*$ denotes the value from the maximal trade that includes trade λ_i^* to agent i.

In our model we introduce parameters, $\delta_{\text{crit}} > 0$, and $\alpha_{\text{bb}} > 0$, and make the following assumptions about the critical value and about the surplus that is available to allocate as discounts to agents:

[A1] the critical value x_{crit} is uniformly distributed about v_i, i.e. $x_{\text{crit}} \sim U(v_i - \delta_{\text{crit}}, v_i + \delta_{\text{crit}})$.

[A2] the available surplus to allocate as discount to agent i is exogenous, equal to $\alpha_{\text{bb}}\delta_{\text{crit}}$.

[A3] the exchange implements *ex ante* budget-balance.

Agent i does not know the actual value of, x_{crit}, only the distributional information. This models an agent with uncertain information about the bids and asks of other agents. Notice that we do not model the equilibrium effect of an agent's own bidding strategy on the distribution over the critical value x_{crit}.

The available surplus is assumed independent of the bids of agent i, assumption [A2]. This assumption about fixed surplus is reasonable within a model of an exchange with *ex ante* budget-balance, assumption [A3], in which the average surplus is distributed back to agents as discounts every time the exchange is cleared. The parameters within a particular payment model (e.g. the Threshold parameter) are adjusted to achieve budget-balance as an agent's bid price changes. A change in agent i's bids has an expected change on the payments made to agent i, based on the distribution over critical value, x_{crit}.

In Section 5.2.1 we determine the optimal bidding strategy of agent i in each payment rule, as a function of the parameter of the rule, such as C_t in Threshold, or μ in Fractional. In Section 5.2.2 we consider the expected discount to the agent with this optimal bidding strategy, and then compute the optimal manipulation under each payment rule when the parameters are selected to give ex ante budget-balance. The analysis leads to a relationship between the available surplus, $\alpha_{\text{bb}}\delta_{\text{crit}}$, and the manipulation, $\theta_i^*(\alpha_{\text{bb}})$, selected by the agent in each rule.

Throughout this section we choose to illustrate the analysis with respect to the Threshold rule. We omit the analysis of the other rules, and just give the results of the analysis in tabular form.

5.2.1 Determining the Optimal Manipulation

Manipulation by an agent has two effects on the expected utility for an agent: (i) the probability of the adjusted bid being accepted decreases, and (ii) the total utility if the bid is accepted can go up because the agent's payment might be reduced. Payment rules change (ii) but not (i), and in turn effect agents' bids and the efficiency of the exchange.

In Figure 9 we plot the *utility* for a particular bid, $b_i = v_i - \theta$, as the critical value x_{crit} varies. Recall that an agent's utility $u_i(\theta)$ for a bid $b_i = v_i - \theta$ is

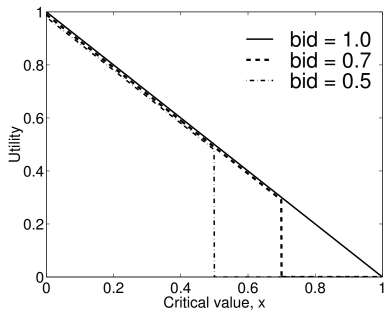
$$u_i(\theta) = \begin{cases} v_i - p, & \text{if } v_i - \theta > x_{\text{crit}} \\ 0, & \text{otherwise} \end{cases}$$

where p is the payment by the agent to the exchange when its bid is accepted. Each sub plot in Figure 9 is for a different payment scheme, and each line within a sub plot corresponds to the utility to an agent for a different bid as the critical value x_{crit} changes. We assume in this plot that the agent's value is $v_i = 1$, and consider critical values $0 \leq x_{\text{crit}} \leq 1$ and manipulation levels $\theta \in \{0, 0.3, 0.5\}$. The Threshold parameter, $C_t = 0.4$, and the Fractional parameter, $\mu = 0.5$. Although not plotted here, the curves for Equal are similar to the No-Discount case (except shifted higher in utility by a constant amount), and Large is similar to Threshold.

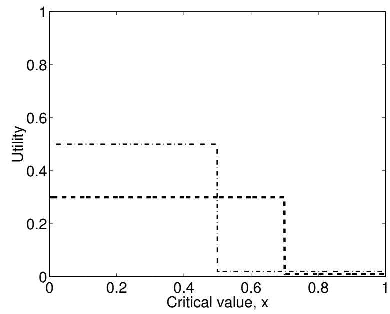
Let $\Delta_{\text{vick},i}(b_i)$ denote the Vickrey discount to agent i given bid b_i.

$$\Delta_{\text{vick},i}(b_i) = \begin{cases} b_i - x_{\text{crit}}, & \text{if } b_i \geq x_{\text{crit}} \\ 0, & \text{otherwise} \end{cases}$$

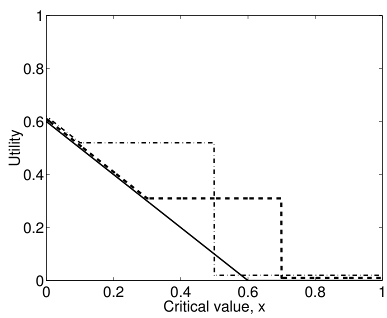
In Threshold the discount rule is $\Delta_i = \max(0, \Delta_{\text{vick},i}(b_i) - C_t)$, for parameter C_t, where $\Delta_{\text{vick},i}(b_i)$ is the discount computed using the Vickrey payment rule. The agent's payment



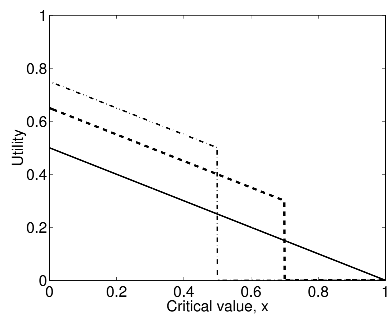
(a) Vickrey



(b) No-Discount



(c) Threshold



(d) Fractional

Figure 9: Utility of bids $b_i = v_i - \theta$ with $\theta \in \{0, 0.3, 0.5\}$, $v = 1$, as the critical value x_{crit} varies between 0 and 1. $C_t = 0.4$ in Threshold, and $\mu = 0.5$ in Fractional.

is 0 if $b_i < x_{\text{crit}}$, because it does not trade. On the other hand, if $b_i \geq x_{\text{crit}}$, we know that $\Delta_{\text{vick},i}(b_i) = b_i - x_{\text{crit}}$, and the agent's payment is $b_i - 0$ if $b_i \leq x_{\text{crit}} + C_t$, or $b_i - (b_i - x_{\text{crit}} - C_t) = x_{\text{crit}} + C_t$ if $b_i > x_{\text{crit}} + C_t$. Putting this together, the agent's utility, $u_i(\theta, C_t, x_{\text{crit}})$, given bid $b_i = v_i - \theta$ is:

$$u_i(\theta, C_t, x_{\text{crit}}) = \begin{cases} v_i - (x_{\text{crit}} + C_t), & \text{if } v_i - \theta \geq x_{\text{crit}} + C_t \\ v_i - (v_i - \theta), & \text{if } x_{\text{crit}} + C_t > v_i - \theta \geq x_{\text{crit}} \\ 0, & \text{otherwise} \end{cases}$$

Looking at Figure 9, in the Vickrey scheme a lower bid reduces the agent's expected utility because it decreases the probability of success without increasing the utility of a successful bid. In comparison, with No Discount, the agent gains utility on all successful bids by the amount of deviation from truthful bidding. In the Threshold scheme a lower bid only reduces the price paid for a limited range of critical values (closer than C_t to the bid price), while in the Fractional scheme a lower bid reduces the price paid on all successful bids (but by less than in the No Discount scheme).

Continuing, we compute the expected utility, $\mathrm{E}u_i(\theta, C)$, in each payment rule as a function of θ and the parameter C in the rule (μ in Fractional). By assumption [A1], the critical value x_{crit} is uniformly distributed about v_i. The expected utility for each level of manipulation θ can be computed as the area under a particular curve in a plot like Figure 9. In turn, the expected-utility maximizing bid corresponds to the curve with maximum area.

Returning to Threshold, assume that $C_t \leq \delta_{\text{crit}}$, so that the agent will receive a discount for some choice of $\theta < \delta_{\text{crit}}$. Let $f(x)$ denote the probability density function for critical value x_{crit}, and consider three cases:

Case 1 ($0 \leq \theta \leq \delta_{\text{crit}} - C_t$) In this case, for almost truthful bids, the agent's Vickrey discount can be both above and below the Threshold parameter C_t, depending on x_{crit}:

$$\mathrm{E}u_i(\theta, C_t) =$$

$$\int_{x_{\text{crit}}=v-\delta_{\text{crit}}}^{v-\theta-C_t} [v-(x_{\text{crit}}+C_t)] f(x_{\text{crit}}) \mathrm{d}x_{\text{crit}} + \int_{x_{\text{crit}}=v-\theta-C_t}^{v-\theta} [v-(v-\theta)] f(x_{\text{crit}}) \mathrm{d}x_{\text{crit}} + \int_{x_{\text{crit}}=v-\theta}^{v+\delta_{\text{crit}}} 0 f(x_{\text{crit}}) \mathrm{d}x_{\text{crit}}$$

$$= \frac{(\delta_{\text{crit}}-\theta-C_t)}{2\delta_{\text{crit}}}(v-C_t) - \frac{1}{4\delta_{\text{crit}}}\left[(v-\theta-C_t)^2 - (v-\delta_{\text{crit}})^2\right] + \frac{\theta}{2\delta_{\text{crit}}}(\theta + C_t - \theta)$$

Case 2 ($\delta_{\text{crit}} - C_t < \theta \leq \delta_{\text{crit}}$) In this case the agent's bid is far enough away from truthful that its Vickrey discount is never above the Threshold parameter C_t.

$$\mathrm{E}u_i(\theta, C_t) = \int_{x_{\text{crit}}=v-\delta_{\text{crit}}}^{v-\theta} [v-(v-\theta)] f(x_{\text{crit}}) \mathrm{d}x_{\text{crit}} = \theta(\delta_{\text{crit}} - \theta)/2\delta_{\text{crit}}$$

Case 3 ($\delta_{\text{crit}} < \theta$) In this case the agent's bid is never greater than the critical value, and $\mathrm{E}u_i(\theta, C_t) = 0$.

Next, we compute the agent's *expected-utility maximizing* bidding strategy, denoted $\theta_i^*(C_t)$. Agent i has distributional information about x_{crit}. Differentiation of $\mathrm{E}u_i(\theta, C_t)$ w.r.t. θ, and then case analysis, we have:

$$\theta_i^*(C_t) = \min\left[C_t,\ \delta_{\text{crit}}/2\right]$$

As special cases, when $C_t = \delta_{\text{crit}}$ (no discount), we have $\theta_i^*(C) = \delta_{\text{crit}}/2$, and with $C_t = 0$ (Vickrey discount), we have $\theta_i^*(C) = 0$.

In Table 4 we present the expressions for the expected utility maximizing strategy in each payment rule, as a function of the parameter δ_{crit} in the distribution of the critical value, and the parameterization of each rule. It is useful to confirm that all expressions reduce to that for the Vickrey and No-Discount rules at extreme parameter values; e.g. $\mu^* \in \{0, 1\}$ in Fractional, $C_t^* \in \{0, \delta_{\text{crit}}/2\}$ in Threshold, etc.

Rule	Optimal Manipulation, $\theta_i^*(C)$	Expected Discount, $\mathrm{E}\Delta_i(\theta^*(C), C)$
No-Discount	$\delta_{\text{crit}}/2$	0
Vickrey	0	$\delta_{\text{crit}}/4$
Fractional	$\max\left[0, \left(\frac{1-\mu}{2-\mu}\right)\delta_{\text{crit}}\right]$	$\min\left[\delta_{\text{crit}}/4, \frac{\delta_{\text{crit}}\mu}{4(2-\mu)^2}\right]$
Threshold	$\min\left[C_t, \delta_{\text{crit}}/2\right]$	$\max\left[0, \frac{(\delta_{\text{crit}}-2C_t)^2}{4\delta_{\text{crit}}}\right]$
Equal	$\frac{\delta_{\text{crit}}-D}{2}$	$\frac{D(\delta_{\text{crit}}+D)}{4\delta_{\text{crit}}}$
Small	$\max\left[0, \min\left(\delta_{\text{crit}}/2, \delta_{\text{crit}} - C_s\right)\right]$	$\min\left[\delta_{\text{crit}}/4, C_s^2/4\delta_{\text{crit}}\right]$
Large	0, if $C_l \leq \delta_{\text{crit}}/\sqrt{2}$	$-C_l^2/4\delta_{\text{crit}} + \delta_{\text{crit}}/4$, if $C_l \leq \delta_{\text{crit}}/\sqrt{2}$
	$\delta_{\text{crit}}/2$, otherwise	0, otherwise
Reverse	$\max\left[0, \frac{\delta_{\text{crit}}-C_r}{2}\right]$	$\min\left[\delta_{\text{crit}}/4, C_r/4\right]$

Table 4: Manipulation and Expected Discount in the Analytic Model.

5.2.2 Balancing Expected Discount with Surplus

In order to compare the optimal manipulation that is selected by an agent in each payment rule, we compute the expected discount that is allocated to the agent in each rule, and carefully select rule parameters to equalize the expected discount across the rules. Interestingly, we show that the rules have quite different effects on the agent's optimal level of manipulation, even with the same expected discount to an agent.

First, we must compute the *expected discount* allocated to agent i in each payment scheme, as the parameter in the scheme varies, and assuming that the agent follows its optimal bidding strategy (as computed in Table 4). Returning to the Threshold example, given bid θ, critical value x_{crit}, and parameter C_t, the discount in the Threshold rule to agent i is 0 when $b_i < x_{\text{crit}}$, and $\max(0, \Delta_{\text{vick},i}(b_i) - C_t)$ otherwise. Combining, we can write the discount $\Delta_i(\theta, C_t, x_{\text{crit}})$ as:

$$\Delta_i(\theta, C_t, x_{\text{crit}}) = \max[0, v - \theta - (x_{\text{crit}} + C)]$$

Continuing, we can compute the *expected* discount, $\mathrm{E}\Delta_i(\theta, C_t)$, wrt the distribution over the critical value x_{crit}. First, in the case that $\theta \leq \delta_{\text{crit}} - C$, we have:

$$\begin{aligned}\mathrm{E}\Delta_i(\theta, C_t) &= \int_{x_{\text{crit}}=v-\delta_{\text{crit}}}^{v-\theta-C_t} [v - \theta - (x_{\text{crit}} + C_t)] f(x_{\text{crit}}) \mathrm{d}x_{\text{crit}} \\ &= (v - \theta - C_t)\frac{(\delta_{\text{crit}} - \theta - C_t)}{2\delta_{\text{crit}}} - \frac{1}{4\delta_{\text{crit}}}\left[(v - \theta - C_t)^2 - (v - \delta_{\text{crit}})^2\right]\end{aligned}$$

In the case of $\theta > \delta_{\text{crit}} - C_t$, we have $\mathrm{E}\Delta_i(\theta, C_t) = 0$.

We can now substitute for the agent's optimal bidding strategy, $\theta_i^*(C_t)$, to compute the expected discount, $\mathrm{E}\Delta_i(\theta_i^*(C_t), C_t)$, as a function of rule parameter C_t:

$$\mathrm{E}\Delta_i(\theta_i^*(C_t), C_t) = \max\left[0, \frac{(\delta_{\mathrm{crit}} - 2C_t)^2}{4\delta_{\mathrm{crit}}}\right]$$

In Table 4 we present the expected discount in each payment rule, as a function of the parameter in the rule and the parameter δ_{crit} of the distribution over the critical value.

Let us now compare how well each payment rule provides incentives to minimize agent manipulation. Figure 10 plots the expected *gain* in utility (in comparison with truthful bidding), $\mathrm{E}u_i(\theta) - \mathrm{E}u_i(0)$, in each payment rule. The parameters in each rule are carefully tuned to ensure that the expected discount to the agent in each rule, when the agent follows its optimal strategy, is the same. In this case we set the parameters to give the expected discount in each rule equal to $0.1\delta_{\mathrm{crit}}$. Notice that the level of manipulation, θ_i^*, that maximizes the agent's gain in utility is smallest in the Threshold scheme for this value of surplus, with Large not far behind.

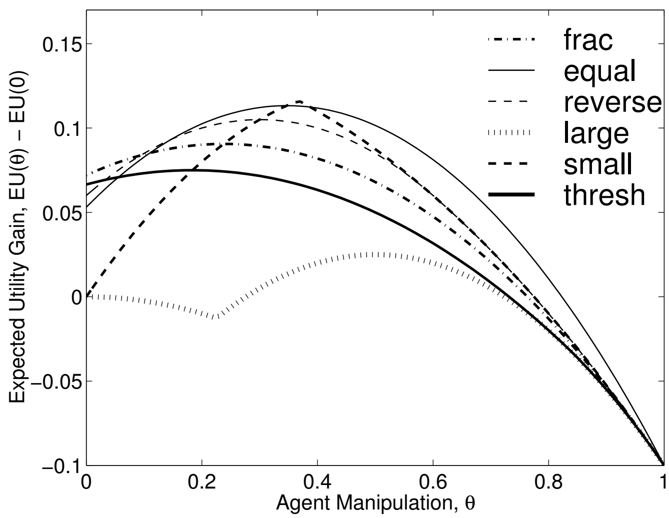


Figure 10: Expected Gain in Utility for different bids $b = v - \theta$ under each payment scheme. The parameters in each rule are carefully set to make each rule allocate the same expected discount, when the agent follows its optimal strategy for that rule. In this case the expected discount is $0.1\delta_{\mathrm{crit}}$ in each rule.

Finally, we can understand the effect that the choice of payment rule has on agent's optimal manipulation levels for different levels of expected discount. By assumption [A2], we assume that there is an average surplus of $\alpha_{\mathrm{bb}}\delta_{\mathrm{crit}}$ available to allocate as discount to agent i, and compare the rules for $\alpha_{\mathrm{bb}} \in [0, 1]$.

Returning again to the Threshold rule, we need to compute the parameter $C_t^*(\alpha_{\mathrm{bb}})$,

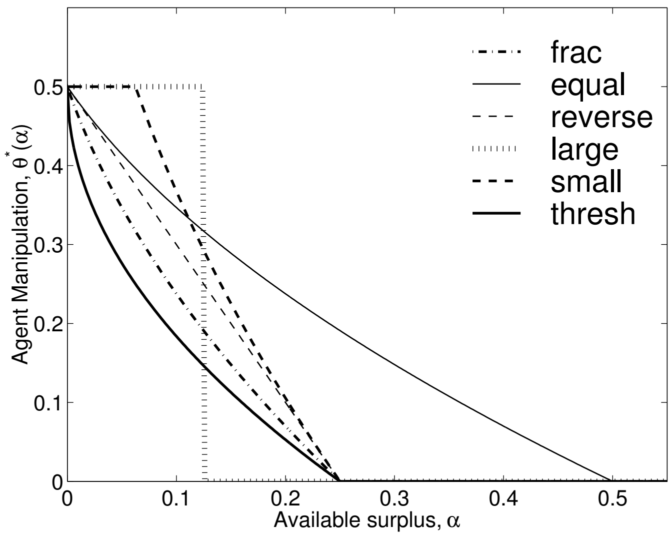


Figure 11: Optimal agent manipulation, $\theta_i^*(\alpha_{\text{bb}})$, (as a proportion of δ_{crit}) under each payment scheme as the amount of available surplus increases from 0 to $\delta_{\text{crit}}/2$ per-agent. The parameters are carefully selected at each α_{bb} to give budget-balance in each rule.

for surplus $\alpha_{\text{bb}}\delta_{\text{crit}}$, that minimizes agent manipulation but ensures budget-balance; i.e. compute the minimal C_t such that $\mathrm{E}\Delta_i(\theta_i^*(C_t), C_t) \leq \alpha_{\text{bb}}\delta_{\text{crit}}$.

By Assumption [A3], we implement *ex ante* budget-balance, such that the discounts balance the surplus on average over the distribution of critical values, x_{crit}. In the case of Threshold, the solution is:

$$C_t^*(\alpha_{\text{bb}}) = \max\left[0, \frac{\delta_{\text{crit}}}{4}\left(2 - \sqrt{16\alpha_{\text{bb}}}\,\right)\right]$$

$C_t^*(\alpha_{\text{bb}})$ is the Threshold parameter that minimizes manipulation and maintains budget-balance. With $\alpha_{\text{bb}} \geq 1/4$, then $C^*(\alpha_{\text{bb}}) = 0$ and the rule implements Vickrey discounts, while for $\alpha_{\text{bb}} = 0$, then $C^*(\alpha_{\text{bb}}) = \delta_{\text{crit}}/2$ and the rule implements no discounts.

Finally, we have a relationship between the expected discount allocated to agent i and the level of manipulation selected by agent i, when the parameter in Threshold is carefully selected to give budget-balance:

$$\begin{aligned}\theta_i^*(\alpha_{\text{bb}}) &= \min[C_t^*(\alpha_{\text{bb}}), \delta_{\text{crit}}/2] \\ &= C_t^*(\alpha_{\text{bb}}) = \max\left[0, \frac{\delta_{\text{crit}}}{4}\left(2 - \sqrt{16\alpha_{\text{bb}}}\,\right)\right]\end{aligned}$$

Plugging in values, if $\alpha_{\text{bb}} \geq 1/4$ then $\theta_i^*(\alpha_{\text{bb}}) = 0$, while if $\alpha_{\text{bb}} = 0$ then $\theta_i^*(\alpha_{\text{bb}}) = \delta_{\text{crit}}/2$.

Figure 11 plots these relationships between agent manipulation and surplus parameter α_{bb} under each payment rule. Notice that Vickrey payments can be implemented with surplus $\geq \delta_{\mathrm{crit}}/4$, so all schemes except Equal and No-Discount prevent manipulation completely for $\alpha_{\mathrm{bb}} \geq 1/4$. For smaller amounts of surplus the market maker is forced to deviate from Vickrey, and move left in the Figure, reducing the level of expected discount that can be paid to each agent. Finally, with no available discount, $\alpha_{\mathrm{bb}} = 0$, no scheme can have a beneficial effect on manipulation and the agent will manipulate by $\delta_{\mathrm{crit}}/2$.

Notice that the simple minded Equal scheme appears to have quite bad incentive properties. In fact, the Threshold method dominates all other schemes in this model except Large. Large has an interesting bad-good phase transition at $\alpha_{\mathrm{bb}} = 1/8$, and can prevent manipulation completely for $1/8 \leq \alpha_{\mathrm{bb}} \leq 1/4$ even though agents with small Vickrey discounts might have benefited from manipulation with hindsight.

Essentially, each payment rule imposes different tradeoffs on an agent between the expected loss from bidding below the critical value and losing a beneficial trade, and the positive effect that a lower bid price can have on an agent's payment. Some rules design the tradeoffs to make manipulation less desirable, for example making it the case that an agent can only hope to reduce its payment in the problems in which manipulation is also more likely to drop the agent's bid price below the critical value.

6 Experimental Analysis

In this section we provide experimental results that compare the manipulation by agents, this time in a more realistic model than the model assumed in the previous section.

The key differences in the experimental analysis are:

- *ex post* budget-balance, with the surplus as computed after the exchange is cleared.
- rule parameters are computed *after* bids and asks are received to implement the optimal discount allocation within each rule.
- we allow agents to adjust their bid and ask prices on multiple trades.

The analysis quantifies the level of manipulation in each scheme for different distributions of problem instances, and also quantifies the *allocative efficiency* in each scheme.

6.1 The Experimental Model

We make the following simplifying assumptions:

[A4] we consider only simple manipulations by $y\%$, where an agent reduces all of its reported values by $y\%$, e.g. shaving bid prices upwards and ask prices downwards.

[A5] we consider only symmetric Nash equilibria in which every agent either manipulates by 0% or y%.

[A6] we assume that agents select the symmetric Nash equilibria which maximizes the expected individual increase in utility from manipulation by y% in comparison with 0%.[12]

[A7] the valuations of all agents are drawn from the same distributions.

[A8] agents are either buyers or sellers.

Notice that by requiring agents to select a static manipulation policy, y^*%, across all problem instances we are making an implicit assumption that agents do not have anything other than distributional information about the bids and asks from agents in any particular instance of the exchange.

The problem instances are generated by adapting distributions Random, Weighted Random, Decay, and Uniform distributions from Sandholm [27], to generate values on bundles instead of bid prices in a combinatorial auction. In each instance we generate values on 100 bundles, in a problem with 50 items, and then distribute the bundles across agents to generate valuation functions for the agents. We consider problems with 5, 10, and 20 agents, and buyers and sellers in proportions (buyer/seller) of { 5/5, 7/3, 2/3, 4/1, 10/10, 15/5 }. We assume that a buyer wants to buy *at most one* of the bundles for which it has value, and that a seller wants to sell *at most one* of the bundles for which it has value. The exchange design allows agents to submit exclusive-or bids on bundles, so that an agent can quite easily represent its truthful valuation function. We assume free-disposal, and clean-up any distributions that are generated that do not satisfy this property. The exchange allows full aggregation, such that a bid can be matched with any number of asks, and an ask can be matched with any number of bids.

For each problem configuration we generate 80 problem instances. For each instance, we consider a discrete number of manipulation levels y%. At each level we compute the average single-agent gain in utility from manipulation by y% in comparison with no manipulation, assuming that every other agent manipulates by y%, with each agent considered in turn. Finally, the single-agent gain in utility is averaged across all agents, and then across all instances for that configuration. The allocative efficiency for each configuration and each level of manipulation is averaged over all instances, and computed for the case that every agent manipulates by y%.

[12]It is not enough to simply measure the absolute utility at different levels of manipulation because even an exchange with Vickrey payments is not safe from collusion. Although it is an optimal strategy for an individual agent in a Vickrey exchange to bid truthfully whatever the bids/asks of other agents, the bids/asks of other agents can have a beneficial effect on its own utility.

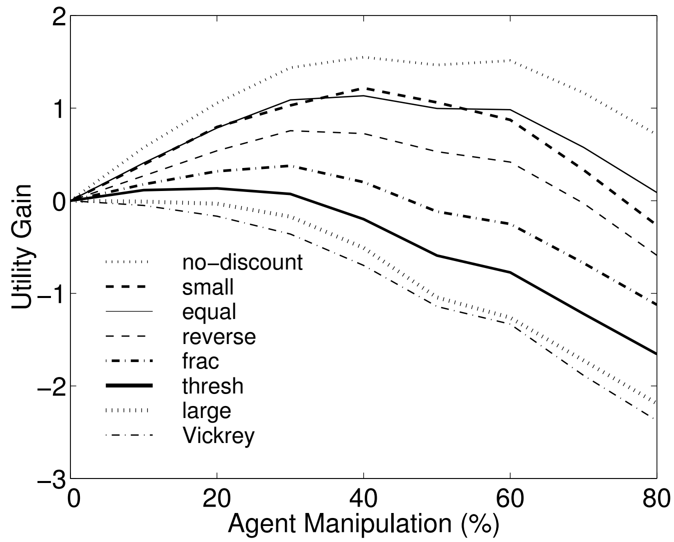


Figure 12: Average Single-Agent Gain in Utility from manipulation by $y\%$ (vs. truthful bidding), in a system in which every other agent manipulates by $y\%$. Problem size: 5 buyers/5 sellers.

6.2 Experimental Results: Manipulation

Figure 12 plots the average single agent gain in utility for different levels of manipulation, $y\%$, under the assumption that every other agent manipulates by $y\%$, for a problem with 5 buyers and 5 sellers.

By assumption [A6], we compute the symmetric Nash equilibrium for a particular payment rule as the level of manipulation, $y^*\%$, that coincides with the peak of a plot such as that in Figure 12. In this case, for the 5 buyers/5 sellers problem, the equilibrium manipulation level in Large and Threshold is less than under the other rules: around 10% and 20% in Large and Threshold, compared with 30%, 40% and 50% in Fractional, Equal and No-Discount. In addition, the amount of utility gain in Large and Threshold is much less than in the other schemes. In addition, notice that Vickrey, Large and Threshold have lower absolute *values* of utility gain across the range of possible levels of manipulation, and that the maximal individual agent gain from manipulation is lower than in other schemes.

By way of comparison, Figure 13 plots the average single-agent utility gain from manipulation in an exchange with more buyers than sellers, in this case 12 buyers and 4 sellers. Although the same basic observations hold, with the equilibrium manipulation level in the Threshold and Large schemes less than in the other schemes, the optimal level of manipulation appears to be higher than in the 5 buyer/ 5 seller problem in Figure 12. In fact this effect might not be too important because the absolute gain from manipulation of Threshold and Large is smaller in this experiment.

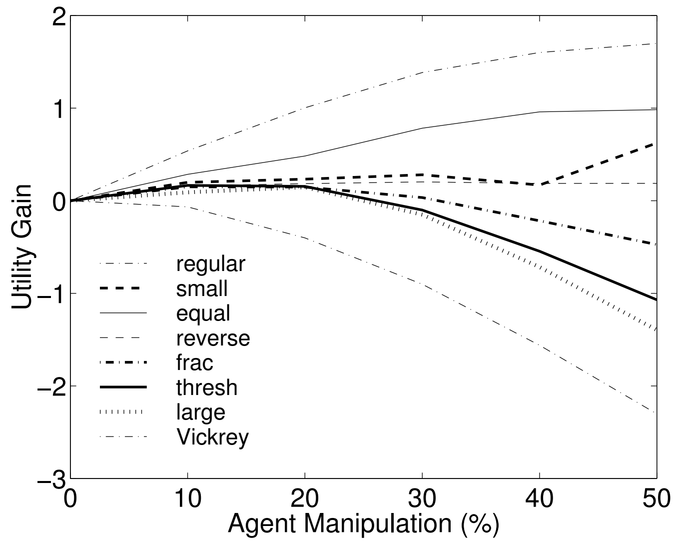


Figure 13: Average Single-Agent Gain in Utility from manipulation by $y\%$ (vs. truthful bidding), in a system in which every other agent manipulates by $y\%$. Problem size: 12 buyers/4 sellers.

In Table 5 we summarize the experimental manipulation results across all problem configurations. We tabulate the average optimal degree of manipulation by agents in each scheme. Over all problems the Large and Threshold schemes perform quite well, with around 20% manipulation; this is in comparison, for example, with approaching 50% manipulation in the Equal, Small, and No Discount schemes. We also compute the average utility gain in each scheme from manipulation at 10%, 20%, and 30%; again Large and Threshold do well under this metric.

6.3 Experimental Results: Efficiency

Figure 14 (a) plots the allocative efficiency of the exchange at different levels of manipulation, averaged across all problem configurations. Notice that the allocative efficiency falls almost linearly (and even point-for-point) with manipulation, from 100% at 0% manipulation to 20% at 80% manipulation. Table 5 presents the average allocative efficiency in the exchange under each payment rule, at the optimal manipulation equilibrium. Threshold and Large achieve greater than 85% allocative-efficiency, while Equal Small and No Discount manage only around 60% efficiency. The efficiency effect is quite significant at these manipulation levels.

It is also interesting to consider how much of a problem budget-balance is with the Vickrey mechanism in these problems. Figure 14 (b) plots the degree of budget-balance failure with Vickrey payments at different levels of manipulation. The budget deficit with

	No-Discount	Vickrey	Small	Frac
Utility Gain	0.799	-0.195	0.479	0.211
Correlation	0.053	1.0	0.356	0.590
Manipulation, θ^*	48	0	48	32
Efficiency (%)	58	100	58	78
	Threshold	Equal	Large	Reverse
Utility Gain	0.110	0.516	0.029	0.337
Correlation	0.543	0.356	0.176	0.522
Manipulation, θ^*	22	46	18	44
Efficiency (%)	86	62	88	64

Table 5: Experimental results, averaged across all configurations. *Utility gain* is the average single-agent utility gain from manipulation in each scheme, averaged over manip. 10%, 20%, and 30%. *Correlation* with Vickrey discounts is also computed for manip. 10%, 20% and 30%. *Manipulation* is the average equilibrium level of manipulation selected by agents. *Efficiency* is the average allocative efficiency at that level of manipulation.

Vickrey discounts falls linearly as the level of manipulation increases, but is still a problem even when agents manipulate by 80%.

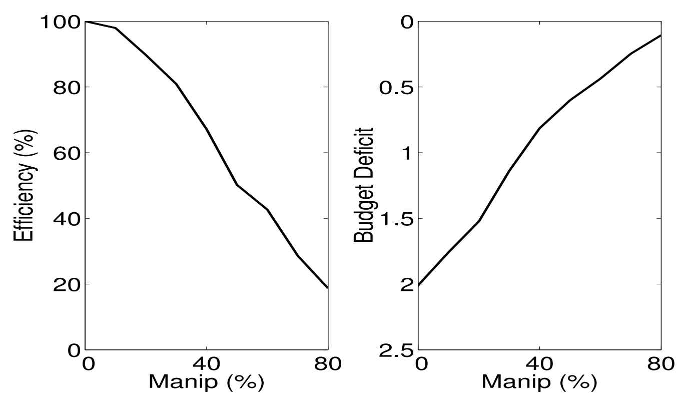


Figure 14: Experimental results, averaged across all problem configurations. (a) Allocative Efficiency; (b) Budget Deficit with Vickrey payments.

We also compare the *correlation* between Vickrey discounts and actual discounts under each scheme in Table 5. While the discounts in Fractional and Threshold are quite well correlated with the Vickrey discounts, notice that the discounts in Large are *not* very well correlated.

7 Discussion

The partial ordering {Large, Threshold } $\succ$ Fractional $\succ$ Reverse $\succ$ {Equal, Small } from the experimental results is remarkably consistent with the results of the simple analytical model that we presented in Section 5.2. Although the Large scheme generates slightly less manipulation and higher allocative efficiency than Threshold in the experimental tests, the Threshold discounts are quite well correlated with the Vickrey discounts while the Large discounts are quite uncorrelated with the Vickrey discounts. This points to the fact that an agent's discount in Large is very sensitive to its bid. We expect Large to be less robust than Threshold in practice because of this all-or-nothing characteristic.

There are a number of observations to make about the geometric properties of the Large and Threshold rules (see Figure 8), in comparison with the other rules. Recall that the critical value is the minimal price that the agent can bid and still execute the same trade when the exchange clears.

1. Rules with flat sections are better than rules without (e.g. Large, Threshold vs. Equal, Fractional)

2. Rules with flat sections for bids that are a long distance from an agent's critical value (e.g. Large, Threshold) perform better than rules with flat sections for bids that are close to an agent's critical value (e.g. Small, Reverse).

Rules with flat sections for bids that are a long distance from an agent's critical value provide discounts to agents with large Vickrey discounts, but no discounts to agents with small Vickrey discounts. Recognizing that a large Vickrey discount indicates that the agent had a large opportunity to reduce its final payment in a No Discount mechanism, this focus on agents with large Vickrey discounts is useful because it positions incentives to compensate agents with (ex post) easy opportunities for manipulation, while providing no incentives to reduce manipulation to agents with hard opportunities.

Returning to the residual-degree-of-manipulation-freedom (RDMF), it is perhaps useful to think of the payment rules in terms of fruit picking. The mechanism has only limited resources (surplus), but can at least see the fruit (opportunities for manipulation). The goal of the mechanism is to make it as hard as possible for agents to pick fruit. The Large mechanism picks the low-hanging fruit, while the Threshold rule just hides them higher in the tree so that they are more difficult to see from the branches (this takes less effort than actually picking the fruit, which is analogous with the agents not being able to see them at all). The agents cannot see the fruit very clearly, and are reluctant to try to pick fruit that might be too hard to pick.

Consider a model of manipulation in which an agent will manipulate by larger amounts, and more frequently, as RDMF increases. We expect that the uncertain information available to agents about the preferences and strategies of other agents will provide a marginal-increasing relationship between the level of manipulation and the RDMF. In this case the optimal method to minimize total manipulation across agents is precisely that of the Threshold rule, to allocate discounts to minimize the maximal RDMF. The optimality of this greedy rule is easy to see because there are decreasing returns in allocating each additional unit of available discount to the same agent. In comparison, if manipulation was linearly related to RDMF, as we might expect in a system with perfect information, then all discount allocation methods would have a similar effect on total manipulation.

In terms of efficiency, the picture is less clear. In general, reducing agent manipulation can only increase allocative-efficiency, but the easy low fruit that are picked in preference of the difficult high fruit would also appear to be the manipulation opportunities less likely to disrupt the efficient trade.

8 Future Work

The most immediate direction for future work is to complete an equilibrium analysis of the most interesting payment rules (e.g. Large, Threshold, Fractional), where the surplus and bids from other agents are endogenous within the model, and computed in Bayesian-Nash equilibrium. We would expect the equilibrium to be closer to truth-revelation for Threshold than for Large. Another interesting approach would be to take an extreme perfect information view, and consider the efficiency effects of each mechanism when each agent has perfect information about the other agents. We would expect a multiplicity of equilibria, with agents bargaining over how to share the surplus.

Interesting extensions include: consider the effect of strategic manipulation through timing of bids and asks; complete a more complete experimental analysis with a richer strategy space; complete a theoretical analysis to establish rules with worst-case optimal performance; examine the effect of solving the WD and pricing problems approximately on the manipulations; and consider randomized payment rules, and payment rules with multiple flat regions.

9 Conclusions

We proposed a family of budget-balanced and individual-rational mechanisms for combinatorial exchanges, that clear the exchange to maximize reported surplus and then allocate discounts to agents to minimize a suitable distance metric to Vickrey payments. The mechanisms are explicitly designed *not* to be incentive-compatible, both to be able to clear the exchange to maximize reported surplus, and also to leverage the bounded-rationality and limited information available to agents, that makes them unable to fully exploit opportunities for manipulation.

The analytical and experimental results both suggest that a simple Threshold rule has useful incentive properties, and provides higher allocative efficiency than other rules. The effect of the Threshold rule is to remove easy opportunities for manipulation, without attempting to provide incentives for truth-revelation to agents with hard opportunities for manipulation.

Finally, we note that the schemes outlined here can also allow a market maker to make a small profit by taking a sliver of budget-balance, or used in combination with a participation charge to move payments closer to Vickrey payments.

10 Acknowledgments

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11 Appendix: Computing the Distance Functions

In what follows we will give details about how to derive the discount (payment) function for the different distance functions of Section 4. The payment functions obtained are summarized in Table 2.

To solve the problem captured by the mathematical model we will apply the Lagrangian relaxation technique. We relax the budget balance constraint by introducing an associated Lagrange multiplier λ and moving the constraint into the objective:

$$z(\lambda) = \min_{\mathbf{\Delta}} \mathbf{L}(\mathbf{\Delta}, \mathbf{\Delta}_{\text{vick}}) + \lambda \left(\sum_i \Delta_i - V^* \right)$$

$$\text{s.t.,} \quad \mathbf{0} \leq \mathbf{\Delta} \leq \mathbf{\Delta}_{\text{vick}}.$$

Note that for a given multiplier both the optimal solution and its value will be functions of the multiplier, we denote this solution by $\mathbf{\Delta}^*(\lambda)$ and its value by $z(\lambda)$. Sometimes it will be more convenient to use a function of λ to parameterize the solution, these are the C's and μ in the various payment rules.

Obviously $z(\lambda)$ is a lower bound on the optimal value of the original mathematical model [PP]. Lagrangian optimization involves maximizing $z(\lambda)$ in λ, that is, obtaining the best possible lower bound $z(\lambda^*)$ for the optimal value of [PP]. Weak duality implies that if the solution $\mathbf{\Delta}^*(\lambda^*)$ satisfies the budget balance constraint then it will also be an optimal solution for the original problem [PP]. In this case $z(\lambda^*)$ is the optimal objective value for [PP].

Thus for a given distance function we will need to do the following:

1. solve the Lagrangian subproblem for a given value of the multiplier to obtain a solution parameterized by λ,
2. find the multiplier value that maximizes $z(\lambda)$, and
3. show that the solution to the Lagrangian subproblem for the optimal multiplier also satisfies the relaxed constraint and thus it is an optimal solution for [PP].

Note that for all but one of the distance functions we considered the optimal solution to [PP] is also unique; that is, there is exactly one value for the Lagrange multiplier where the optimal solution to the Lagrangian subproblem will be budget balanced. This means that for values of the parameter on one side of the optimal multiplier the optimal solution to the Lagrangian subproblem will be budget balanced and for values on the other side of the optimal multiplier the solution will never be budget balanced.

Assumptions about the distribution of the Vickrey discounts $\boldsymbol{\Delta}_{\text{vick}}$ and the proportion of their sum to the available surplus V^* are sometimes needed in the derivations. Therefore, we will assume that all Vickrey discounts are positive and that the agents are indexed so that their Vickrey discounts are decreasing (let I denote the total number of agents who might receive discounts):

$$\Delta_{\text{vick},I} \leq \Delta_{\text{vick},I-1} \leq \ldots \leq \Delta_{\text{vick},2} \leq \Delta_{\text{vick},1}.$$

We add dummy points $\Delta_{\text{vick},0} = \infty$ and $\Delta_{\text{vick},I+1} = 0$ and index the interval $[\Delta_{\text{vick},k+1}, \Delta_{\text{vick},k}]$ by k. (To properly partition $[0, \infty)$ these intervals have to be open on one end. In some cases it does matter which end of the interval is open, we will indicate this at the description of the individual distance functions.) We also assume that the sum of Vickrey discounts across all agents exceeds the available surplus ($\sum_1^I \Delta_{\text{vick},i} > V^*$) since otherwise the exchange could distribute Vickrey discounts to the agents and achieve full incentive compatibility; and that the available surplus is positive ($V^* > 0$).

11.1 L_2 and L_∞ norms

We will discuss the L_2 norm in detail, the L_∞ norm can be handled similarly (notice that the L_∞ norm can be thought of as the limit of the L_k norm as k goes to infinity). For the L_2 norm the Lagrangian subproblem is

$$z(\lambda) = \min_{\Delta} \sum_{i=1}^{I} (\Delta_{\text{vick},i} - \Delta_i)^2 + \lambda(\sum_{i=1}^{I} \Delta_i - V^*)$$

$$\text{s.t.} \quad 0 \leq \Delta_i \leq \Delta_{\text{vick},i} \quad i = 1, \ldots, I.$$

11.1.1 Solving the Lagrangian subproblem for a given multiplier

Note that this problem decomposes into smaller problems for each i:

$$\min_{\Delta_i} (\Delta_{\text{vick},i} - \Delta_i)^2 + \lambda \Delta_i$$

$$\text{s.t.} \quad 0 \leq \Delta_i \leq \Delta_{\text{vick},i}.$$

From the first order condition

$$-2(\Delta_{\text{vick},i} - \Delta_i) + \lambda = 0$$

and the bounds on the variable we obtain

$$\Delta_i^*(\lambda) = \max(0, \Delta_{\text{vick},i} - \lambda/2)$$

The second derivative is positive thus the objective function is convex.

Let us introduce C_t for $\lambda/2$, thus

$$\Delta_i^*(C_t) = \max(0, \Delta_{\text{vick},i} - C_t), \quad i = 1, \ldots, I \qquad \text{(Threshold)}$$

Note that if C_t falls into interval k agents $i = 1, \ldots, k$ will receive discounts $\Delta_i^*(C_t) = \Delta_{\text{vick},i} - C_t$ while agents $i = k+1, \ldots, I$ will not receive any discounts.

11.1.2 Finding the best multiplier value

Substituting $\Delta_i^*(C_t)$ into $z(C_t)$, the function in interval k is

$$z^k(C_t) = \sum_{i=k+1}^{I} (\Delta_{\text{vick},i})^2 + 2C_t \left(\sum_{i=1}^{k} \Delta_{\text{vick},i} - V^* \right) - kC_t^2.$$

Observe that the $z(C_t)$ function, which is pieced together from the $z^k(C_t)$-s, is continuous everywhere (it is continuous within each interval and $z^k(\Delta_{\text{vick},k}) = z^{k-1}(\Delta_{\text{vick},k})$). The first derivative of the function in interval k is

$$(z^k)'(C_t) = -2kC_t + 2 \left(\sum_{i=1}^{k} \Delta_{\text{vick},i} - V^* \right),$$

thus $z'(C_t)$, which is pieced together from the $(z^k)'(C_t)$-s, is a continuous everywhere. Also, $z'(C_t)$ is a monotone decreasing function. For a very small value of the parameter

$$z'(\epsilon) = (z^I)'(\epsilon) = -2I\epsilon + 2(\sum_{i=1}^{I} \Delta_{\text{vick},i} - V^*) > 0$$

since the sum of the Vickrey discounts exceeds the available surplus. For any parameter value in interval 0

$$z'(C_t) = (z^0)'(C_t) = -2V^* < 0.$$

Therefore, because of the continuity of the first derivative, there exists an interval K and a unique point C_t^* within this interval so that $z'(C_t^*) = (z^K)'(C_t^*) = 0$; that is,

$$\Delta_{\text{vick},K+1} \leq C_t^* = \frac{\sum_{i=1}^K \Delta_{\text{vick},i} - V^*}{K} \leq \Delta_{\text{vick},K}.$$

11.1.3 Finding the overall best solution

We show that budget balance holds for $\mathbf{\Delta}^*(C_t^*)$, the solution to the Lagrangian subproblem with the best parameter C_t^* computed above. The sum of discounts for parameter C_t^* (which falls into interval K) is

$$\sum_{i=1}^{K}(\Delta_{\text{vick},i} - C_t^*) + \sum_{i=K+1}^{I} 0 = \sum_{i=1}^{K} \Delta_{\text{vick},i} - KC_t^* = V^*,$$

the last equality implied by the definition of C_t^*. This implies that $\mathbf{\Delta}^*(C_t^*)$ is an optimal solution to [PP] of objective value

$$z^K(C_t^*) = \sum_{i=K+1}^{I} (\Delta_{\text{vick},i})^2 + \frac{1}{K}\left(\sum_{i=1}^{K} \Delta_{\text{vick},i} - V^*\right)^2.$$

11.2 The L_{RE} distance functions

For the L_{RE} distance function the Lagrangian subproblem is

$$z(\lambda) = \min_{\Delta} \sum_{i=1}^{I} \frac{(\Delta_{\text{vick},i} - \Delta_i)}{\Delta_{\text{vick},i}} + \lambda(\sum_{i=1}^{I} \Delta_i - V^*)$$

$$\text{s.t.} \quad 0 \leq \Delta_i \leq \Delta_{\text{vick},i} \quad i = 1, \ldots, I.$$

11.2.1 Solving the Lagrangian subproblem for a given multiplier

Note that this problem decomposes into smaller problems for each i:

$$\min_{\Delta_i} 1 + \Delta_i \left(\lambda - \frac{1}{\Delta_{\text{vick},i}}\right)$$

$$\text{s.t.} \quad 0 \leq \Delta_i \leq \Delta_{\text{vick},i}.$$

The minimum is attained at zero if the coefficient of the variable is positive, and at the upper bound of the variable if the coefficient is negative. When the coefficient is zero the variable could take any value, we choose to set it to zero in this case. Let us introduce C_s for $1/\lambda$, thus

$$\Delta_i^*(C_s) = \Delta_{\text{vick},i} \ \text{ if } \ C_s > \Delta_{\text{vick},i}, \ 0 \ \text{ otherwise} \qquad \text{(Small)}$$

Note that if C_s falls into the k^{th} interval $(\Delta_{\text{vick},k+1}, \Delta_{\text{vick},k}]$ then agents $i = 1, \ldots, k$ will not receive any discounts while agents $i = k+1, \ldots, I$ will receive their Vickrey discounts.

11.2.2 Finding the best multiplier value

Substituting $\Delta_i^*(C_s)$ into $z(C_s)$ the function in interval k is

$$z^k(C_s) = k + \frac{1}{C_s}\left(\sum_{i=k+1}^{I} \Delta_{\text{vick},i} - V^*\right).$$

It is easy to see that $z(C_s)$ is continuous. The first derivative of the function in interval k is

$$(z^k)'(C_s) = -\frac{1}{C^2}\left(\sum_{i=k+1}^{I} \Delta_{\text{vick},i} - V^*\right),$$

a monotone decreasing function (but not continuous in the breakpoints). Note that $\sum_{i=k+1}^{I} \Delta_{\text{vick},i} - V^*$ is a constant within each interval and it "jumps" by $\Delta_{\text{vick},k}$ from interval k to $k-1$. For $k = I$ the constant is negative since $V^* > 0$, and for $k = 0$ it is positive since the total of the Vickrey discounts exceeds the available surplus. Thus there exists a unique index K so that the constant is positive for intervals $K, \ldots, I$ and negative for intervals $0, \ldots, K-1$. This implies that the first derivative is positive, and therefore $z(C_s)$ is increasing, for $C_s \leq \Delta_{\text{vick},K}$, and that the first derivative is negative, and therefore $z(C_s)$ is decreasing, for $C_s > \Delta_{\text{vick},K}$.

That is, $C_s^* = \Delta_{\text{vick},K}$ for the index K where $\sum_{i=K+1}^{I} \Delta_{\text{vick},i} \leq V^*$ but $\sum_{i=K}^{I} \Delta_{\text{vick},i} > V^*$.

11.2.3 Finding the overall best solution

It is obvious that budget balance holds for $\mathbf{\Delta}^*(C_s^*)$, the solution to the Lagrangian subproblem with the best parameter C_s^* computed above. The sum of discounts for parameter value $C_s^* = \Delta_{\text{vick},K}$ is

$$\sum_{i=1}^{K} 0 + \sum_{i=K+1}^{I} \Delta_{\text{vick},i} \leq V^*$$

by the choice of the index K. This implies that $\mathbf{\Delta}^*(C_s^*)$ is an optimal solution to [PP] of objective value

$$z^K(\Delta_{\text{vick},K}) = K + \frac{1}{\Delta_{\text{vick},K}} \left(\sum_{i=K+1}^{I} \Delta_{\text{vick},i} - V^* \right).$$

11.3 The L_{WE} distance function

For the L_{WE} distance function the Lagrangian subproblem is

$$z(\lambda) = \min_{\Delta} \sum_{i=1}^{I} \Delta_{\text{vick},i} \left(\Delta_{\text{vick},i} - \Delta_i\right) + \lambda(\sum_{i=1}^{I} \Delta_i - V^*)$$

$$\text{s.t.} \;\; 0 \leq \Delta_i \leq \Delta_{\text{vick},i} \;\; i = 1, \ldots, I.$$

11.3.1 Solving the Lagrangian subproblem for a given multiplier

Note that this problem decomposes into smaller problems for each i:

$$\min_{\Delta_i} (\Delta_{\text{vick},i})^2 + \Delta_i(\lambda - \Delta_{\text{vick},i})$$

$$\text{s.t.} \;\; 0 \leq \Delta_i \leq \Delta_{\text{vick},i}.$$

The minimum is attained at zero if the coefficient of the variable is positive, and at the upper bound of the variable if the coefficient is negative. When the coefficient is zero the variable could take any value up to its upper bound, we choose to set it to zero in this case. Let us introduce C_l for λ, thus

$$\Delta_i^*(C_l) = \Delta_{\text{vick},i} \;\text{ if }\; C_l < \Delta_{\text{vick},i}, \; 0 \;\text{ otherwise} \tag{Large}$$

Note that if C_l falls into the k^{th} interval $[\Delta_{\text{vick},k+1}, \Delta_{\text{vick},k})$ then agents $i = k+1, \ldots, I$ will not receive any discounts while agents $i = 1, \ldots, k$ will receive their Vickrey discounts.

11.3.2 Finding the best multiplier value

Substituting $\Delta_i^*(C_l)$ into $z(C_l)$ the function in interval k is

$$z^k(C_l) = \sum_{i=k+1}^{I} (\Delta_{\text{vick},i})^2 + C_l(\sum_{i=1}^{k} \Delta_{\text{vick},i} - V^*).$$

It is easy to see that $z(C_l)$ is continuous. Note that for $k = 0$ the coefficient of C_l is negative since $V^* > 0$, and that for $k = I$ the coefficient is positive since the total of the Vickrey discounts exceeds the available surplus. Therefore, there exists a unique index K so that the constant is non-positive for $k \leq K$ and is strictly positive for $k \geq K+1$. Thus the function $z(C_l)$ is increasing for $C_l < \Delta_{\text{vick},K+1}$ and decreasing for $C_l \geq \Delta_{\text{vick},K+1}$.

That is, the optimum of $z(C_l)$ is attained at $C_l^* = \Delta_{\text{vick},K+1}$. for the index K where $\sum_{i=1}^{K} \Delta_{\text{vick},i} \leq V^*$ but $\sum_{i=1}^{K+1} \Delta_{\text{vick},i} > V^*$.

11.3.3 Finding the overall best solution

It is obvious that budget balance holds for $\boldsymbol{\Delta}^*(C_l^*)$, the solution to the Lagrangian subproblem with the best parameter C_l^* computed above. The sum of discounts for parameter value $C_l^* = \Delta_{\text{vick},K+1}$ is

$$\sum_{i=1}^{K} \Delta_{\text{vick},i} + \sum_{i=K+1}^{I} 0 \leq V^*$$

by the choice of the index K. This implies that $\boldsymbol{\Delta}^*(C_l^*)$ is an optimal solution to [PP] of objective value

$$z^K(\Delta_{\text{vick},K+1}) = \sum_{i=K+1}^{I} (\Delta_{\text{vick},i})^2 + \Delta_{\text{vick},K+1}(\sum_{i=1}^{K} \Delta_{\text{vick},i} - V^*).$$

11.4 The L_{RE2} distance function

For the L_{RE2} distance function the Lagrangian subproblem is

$$z(\lambda) = \min_{\Delta} \sum_{i=1}^{I} \frac{(\Delta_{\text{vick},i} - \Delta_i)^2}{\Delta_{\text{vick},i}} + \lambda(\sum_{i=1}^{I} \Delta_i - V^*)$$

$$\text{s.t.} \quad 0 \leq \Delta_i \leq \Delta_{\text{vick},i} \quad i = 1, \ldots, I.$$

11.4.1 Solving the Lagrangian subproblem for a given multiplier

Note that this problem decomposes into smaller problems for each i:

$$\min_{\Delta_i} \frac{(\Delta_{\text{vick},i} - \Delta_i)^2}{\Delta_{\text{vick},i}} + \lambda \Delta_i$$

$$\text{s.t.} \quad 0 \leq \Delta_i \leq \Delta_{\text{vick},i}.$$

From the first order condition

$$-\frac{2}{\Delta_{\text{vick},i}}(\Delta_{\text{vick},i} - \Delta_i) + \lambda = 0$$

we obtain

$$\Delta_i = (1 - \frac{\lambda}{2})\Delta_{\text{vick},i}.$$

Let us introduce μ for $1 - \lambda/2$. Thus

$$\Delta_i^*(\mu) = \mu\Delta_{\text{vick},i} \ \text{ if } 0 \le \mu \le 1, \ 0 \ \text{ if } \mu < 0 \ \text{ and } \Delta_{\text{vick},i} \ \text{ if } \mu > 1 \tag{Fractional}$$

That is, if μ is between zero and one, all agents receive discounts proportional to their Vickrey discounts.

11.4.2 Finding the best multiplier value

Substituting $\Delta_i^*(\mu)$ into $z(\mu)$ the function for $\mu < 0$ is

$$z^{<0}(\mu) = \sum_{i=1}^{I} \Delta_{\text{vick},i} - 2(1-\mu)V^*,$$

an increasing function since the coefficient of μ is positive. On the other hand, for $\mu > 1$,

$$z^{>1}(\mu) = 2(1-\mu)(\sum_{i=1}^{I} \Delta_{\text{vick},i} - V^*),$$

a decreasing function since the coefficient of μ is negative. The function for $0 \le \mu \le 1$ can be written as

$$z^{[0,1]}(\mu) = (1-\mu)^2 \sum_{i=1}^{I} \Delta_{\text{vick},i} + 2\mu(1-\mu)(\sum_{i=1}^{I} \Delta_{\text{vick},i} - V^*) =$$

$$= (1-\mu^2) \sum_{i=1}^{I} \Delta_{\text{vick},i} - 2(1-\mu)V^*.$$

It is easy to check that $z(\mu)$ is continuous. First order conditions for $z^{[0,1]}$ imply

$$-2\mu \sum_{i=1}^{I} \Delta_{\text{vick},i} + 2V^* = 0,$$

and since the second derivative is negative,

$$\mu^* = \frac{V^*}{\sum_{i=1}^{I} \Delta_{\text{vick},i}}.$$

μ^* indeed falls into the interval $[0, 1]$ since the sum of Vickrey discounts exceeds the available surplus and both the numerator and denominator are positive.

11.4.3 Finding the overall best solution

Budget balance holds trivially since for the optimal parameter value μ^* the total discount awarded is

$$\sum_{i=1}^{I} \mu^* \Delta_{\text{vick},i} = \frac{V^*}{\sum_{i=1}^{I} \Delta_{\text{vick},i}} \sum_{i=1}^{I} \Delta_{\text{vick},i} = V^*.$$

11.5 The L_Π distance function

Before introducing the Lagrange multiplier we take the logarithm of the L_Π distance function. The optimum after the transformation will be attained at the same value as before the transformation since the logarithm function is continuous and monotone. Note that since the Vickrey discounts are positive the logarithm function can be applied.

Thus after the transformation the Lagrangian subproblem becomes

$$z(\lambda) = \min_{\Delta} \sum_{i=1}^{I} \log(\Delta_{\text{vick},i}) - \sum_{i=1}^{I} \log(\Delta_i) + \lambda(\sum_{i=1}^{I} \Delta_i - V^*)$$

$$\text{s.t.} \quad 0 \le \Delta_i \le \Delta_{\text{vick},i} \quad i = 1, \ldots, I.$$

11.5.1 Solving the Lagrangian subproblem for a given multiplier

After taking the logarithm the problem decomposes into smaller problems for each i:

$$\min_{\Delta_i} \log(\Delta_{\text{vick},i}) - \log(\Delta_i) + \lambda \Delta_i$$

$$\text{s.t.} \quad 0 \le \Delta_i \le \Delta_{\text{vick},i}.$$

Equating the first derivative with zero implies $\Delta_i = 1/\lambda$. Let us introduce C_r for $1/\lambda$. Thus from the first order condition and the bounds on the variables it follows that

$$\Delta_i^*(C_r) = \min(C_r, \Delta_{\text{vick},i}) \qquad \text{(Reverse)}$$

Note that if C_r falls into interval k then agents $i = 1, \ldots, k$ get a discount in the amount of C_r, while agents $i = k+1, \ldots, I$ will receive their Vickrey discounts.

11.5.2 Finding the best multiplier value

Substituting $\Delta_i^*(C_r)$ into $z(C_r)$ the function in interval k is

$$z^k(C_r) = \sum_{i=1}^{k} \log(\Delta_{\text{vick},i}) + k - k \log(C_r) + \frac{1}{C_r} \left(\sum_{i=k+1}^{I} \Delta_{\text{vick},i} - V^* \right).$$

It is easy to check that $z^k(C_r)$ is continuous. The first derivative in interval k is

$$(z^k)'(C_r) = -\frac{k}{C_r} - \frac{1}{C_r^2}\left(\sum_{i=k+1}^{I} \Delta_{\text{vick},i} - V^*\right).$$

$z'(C_r)$ is also a continuous function but not necessarily monotone. For a very small value of the parameter

$$z'(\epsilon) = (z^I)'(\epsilon) = \frac{1}{\epsilon^2}(V^* - \epsilon I) > 0$$

for a small enough ϵ. On the other hand for a large parameter value (that falls into interval zero)

$$z'(C_r) = -\frac{1}{C_r^2}\left(\sum_{i=1}^{I} \Delta_{\text{vick},i} - V^*\right) < 0$$

since the sum of the Vickrey discounts exceeds the available surplus. Therefore, because of the continuity of the first derivative, there exists at least one parameter value where it is zero. Choose C_r^* to be the smallest parameter value (and K to be the corresponding interval) for which the first derivative is zero:

$$C_r^* = -\frac{1}{K}\left(\sum_{i=K+1}^{I} \Delta_{\text{vick},i} - V^*\right).$$

11.5.3 Finding the overall best solution

For the parameter value C_r^* (that falls into the interval K) total of the awarded discounts is

$$KC_r^* + \sum_{i=K+1}^{I} \Delta_{\text{vick},i} = V^*$$

by definition of C_r^*.

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