Dynamic Matching with a Fall-back Option

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Abstract. We study dynamic matching without money when one side of the market is dynamic with arrivals and departures and the other is static and agents have strict preferences over agents on the other side of the market. In enabling stability properties, so that no pair of agents can usefully deviate from the match, we consider the use of a fall-back option where the dynamic agents can be matched, if needed, with a limited number of agents from a separate "reserve" pool. We introduce the GSODAS mechanism, which is truthful for agents on the static side of the market and stable. In simulations, we establish that GSODAS dominates in rank-efficiency a pair of randomized mechanisms that operate without the use of a fall-back option. In addition, we demonstrate good rank-efficiency in comparison to a non-truthful mechanism that employs online stochastic optimization.

1 Introduction

For motivation, we can consider the campus recruitment job market. Companies visit colleges in various time slots during the year, while students are seeking a position throughout the year. In our terms, this is a two-sided matching problem in which the companies are "dynamic" with arrival and departure times while the students are "static" and always present in the market, although perhaps already matched.

Further suppose that students may seek to obtain a better match by strategic misreporting of their preferences over companies, while companies report true preference rankings on students. We can assume this because, it is generally known what skill sets companies want (e.g., which grades, in which kinds of classes, etc.) Student preferences on companies may be predetermined or determined dynamically as companies arrive, as long as preference orderings over earlier companies are not changed by subsequent arrivals. For companies, it is probably easiest to think that their preferences over students are determined upon arrival.

Each company seeks to match with a single student, and a match (if any) must be assigned by the mechanism by the end of the company's time slot. We assume, however, that each company has the opportunity to adopt a "fall-back" option, selected from its own reserve pool of students and providing (if necessary) a match for the company that is just as good as that from the primary matching market. This option can be exercised by the company if a matched student subsequently becomes unavailable because the mechanism later decommits and rematches the student to another company. But the reserve pool should be used in a limited way— we assume that it is more costly, and therefore less desirable for the firm.

Within AI, this work is situated in the subfield of multiagent resource allocation, and for example motivated by an interest in developing AI for crowdsourcing and tasksourcing markets [9, 11]. For the present model, we have workers on one side that seek a match with a new task (e.g., every week a match is formed for the subsequent week) and an uncertain and dynamic supply of tasks, each with preferences over workers and requiring a match to be assigned for the next week by its own deadline.

For expositional purposes only, we refer to the static and strategic side of the market as *Men* and the dynamic and truthful side of the market as *Women*. A constraint imposed by the dynamics of the problem is that the match to a woman must be made before her departure (although with a chance to decommit from this and use a fall back option.) Each man insists on receiving a match only by the final time period, beyond which no additional women will arrive.

We introduce the *GSODAS* (Generalized Online Deferred Acceptance with Substitutes) mechanism, which makes use of the fall-back option, also referred to as *substitute* agents. GSODAS is dominant-strategy truthful for static agents and also *stable*, such that no man-woman pair would prefer to deviate from their respective matches and re-match between themselves. Such a man-woman pair is a *blocking pair.* The blocking pairs includes a man-woman pair where the woman has matched with a fall-back option and insists that for any such pair, that the woman prefers the fall-back option (assumed equivalent in rank-preference to her original match) or the man prefers his match. The number of fall-back options required by GSODAS is worst-case optimal across online mechanisms that provide stability.

We compare the match quality from GSODAS with two randomized, truthful matching mechanisms that operate without using a fall-back option. For match quality, we consider both the *stability* (measuring the average number of men that are in at least one blocking pair) and the *rank-efficiency* of the mechanisms. The rank efficiency measures the average preference rank order achieved by agents in the match, with a rank-order of 1 for most-preferred and n for least-preferred, where there are n agents on each side of the market. For this, we interpret the rank-order for a woman matched with a substitute as equivalent to that for the man with which she was first matched, but ignore the substitute himself in determining rank efficiency. In addition, the preference of any man that goes unmatched is accounted as a rank of $n + 1$.

For a rank-efficiency baseline, we also consider the performance of a non-truthful algorithm, namely, CONSENSUS, that employs online stochastic optimization in determining

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dynamic matches. This provides a strong, baseline target for rank-efficiency. In simulation, we demonstrate that GSODAS has rank-efficiency better than the randomized mechanisms but dominated by CONSENSUS, while the randomized mechanisms and also CONSENSUS also suffer from poor stability and many blocking pairs. GSODAS requires on average around 20% of the women to be matched with fall-back options for two period problems, increasing to an average of 30% of the men for longer 12-period problems (in which the women are present in the market for around 3-4 periods.) The most compelling direction for future work is to find an appropriate relaxation of stability for dynamic problems, and look to see whether this can provide a significant reduction in the use of substitutes.

Related Work. The classic matching algorithm is the deferred acceptance algorithm [4]. This algorithm is strategyproof for one-side of the market and produces a stable match with respect to reported preferences. Moreover, there does not exist a stable matching mechanism that is strategyproof for all agents [8]. We are only aware of one other paper on dynamic matching with incentive and stability considerations: Compte and Jehiel [3] consider a different dynamic matching problem to the one studied here, with a static population but agents that experience a preference shock, and impose an individual-rationality constraint across periods so that no agent becomes worse off as the match changes in response to a shock. The authors demonstrate how to modify the deferred acceptance algorithm to their problem. For more background on the matching literature, readers are referred to a survey by Sönmez and Ünver [10]. Within computer science, Karp *et al.* [6] consider the algorithmic problem of online matching, but without strategic considerations. Awasthi and Sandholm [1] consider a dynamic kidney exchange problem, but for a satisficing (rather than strict preference) model and without consideration of incentive or stability constraints. Parkes [7] provides a survey of dynamic auction mechanisms with money.

2 Preliminaries

Consider a market with n men (set M) and n women (set W). The men are static and the women are dynamic, with woman $i \in W$ having arrival a_i and departure d_i, with $a_i, d_i \in \{1, \ldots, T\}$ where T is the number of time periods. Each agent has a strict preference profile $\succ_i$ on agents on the other side of the market, and prefers to be matched than unmatched. We write $w_1 \succ_m w_2$ to indicate a strict preference by man m for woman w_1 over woman w_2. A match to a man can be made in any of the T periods, and preferences may be determined dynamically as women arrive as long as the preference rank on earlier arrivals is unchanged. Similarly, we write $m_1 \succ_w m_2$ to denote a preference by woman w for man m_1 over m_2. For a woman, a match (if any) must be made between a_i and d_i and preferences must be well-defined upon arrival.

Let $M(t)$ and $W(t)$ denote the set of men and women available for matching in period t. Let $AW(t)$ denote the set of women to arrive in t, $DW(t)$ the set of women to depart in t, and $W'(t)$ the set of women yet to arrive. Let μ denote a match, with $\mu(m) \in W \cup \{\phi\}$ the match to man m and $\mu(w) \in M \cup \{\phi\}$ the match to woman w, with $\mu(i) = \phi$ to indicate that agent i is unmatched. A woman is available for matching while present, and a match $\mu(w) \neq \phi$ to a woman must be finalized by period d_i. Upon the departure of woman w with $\mu(w) \neq \phi$, then the matched man $\mu(w) \in M$ ordinarily becomes unavailable for matching and $M(t)$ is updated. On the other hand, when we allow for a fall-back option the mechanism may decommit from the match $\mu(w)$ and allow a man to re-match.

For static settings, Gale-Shapley's deferred-acceptance (DA) algorithm yields a stable matching. In this paper we adopt the *man-proposing* DA algorithm as a building block:

Definition 1. *Man-proposing DA.* Each man proposes to his most preferred woman. Each woman keeps the best match and rejects other men. All rejected men then propose to their next preferred woman. The procedure continues until there are no more rejections.

We denote $DA(M, W)$ as male proposing DA with set of men M and set of women W. The DA algorithm terminates in a finite number of steps because every man proposes to a finite number of women.

Let $\succ = (\succ_i)_{i \in M \cup W}$. We also write $\succ = (\succ_i, \succ_{-i})$, where $\succ_{-i}$ denote the preferences of all the agents except i. Let $\rho = \{(a_i, d_i) : i \in W\}$ denote the arrival and departure periods of the women. An *online matching mechanism* f selects a matching $\mu = f(\succ, \rho)$. To be feasible, we require that $f(\succ, \rho)$ is invariant to information about later arrivals, so that $\mu(w)$ is invariant to preferences of men about women w' to arrive after w departs or to the preferences, arrival or departure times of later arrivals w'. In particular, $\mu(w)$ must be determined by period d_i at which a woman departs.

Definition 2. Online mechanism f is *truthful* (or *strategyproof*) for men if for each man m, for all arrival-departure schedules ρ, and for all preferences $\prec_{-m}$ except m,

$$\mu'(m) \not\succ \mu(m),$$

where $\mu' = f(\succ'_m, \succ_{-m}, \rho)$.

In evaluating the performance of a mechanism, we follow Budish and Cantillon [2] and assume risk neutral agents with a constant difference in utility across the matches that are adjacent in their preference list. The rank of an agent i for a matching μ, written $rank_i(\mu)$, is the rank order of the agent with whom he or she is matched. A match by i with the most-preferred agent in $\succ_i$ receives rank order 1 and with the least-preferred receives rank order n. If $\mu(i) = \phi$ then the rank-order is $n + 1$. Based on this, the rank of a matching μ is $rank(\mu) = \frac{1}{2n} \sum_{i \in M \cup W} rank_i(\mu)$.

To define the rank-efficiency of a mechanism we assume a distribution function Φ on $(\succ, \rho)$ and compute the expected rank over the induced distribution on matches:

Definition 3. The rank-efficiency of an online mechanism f, given distribution function Φ, is

$$rank^f = \mathbb{E}_{(\succ, \rho) \sim \Phi}[rank(f(\succ, \rho))].$$

To gain some intuition for the dynamic matching problem, we can consider simply running a man-proposing DA on unmatched men and women in the system whenever one or more women departs. As well as fixing the match for any such departing woman, it also sets the match for any man matched to a departing woman. The set of men still available for matching in the future is updated.

Example 2.1. *Consider $M = \{m_1, m_2, m_3\}$ and $W = \{w_1, w_2, w_3\}$. Suppose the preferences and arrival/departure periods are as follows:*

$$m_1 : w_3 \succ_{m_1} w_1 \succ_{m_1} w_2$$
$$m_2 : w_2 \succ_{m_2} w_1 \succ_{m_2} w_3$$
$$m_3 : w_1 \succ_{m_3} w_2 \succ_{m_3} w_3$$
$$w_1 : m_1 \succ_{w_1} m_2 \succ_{w_1} m_3, a_{w_1} = 1, d_{w_1} = 1$$
$$w_2 : m_1 \succ_{w_2} m_2 \succ_{w_2} m_3, a_{w_2} = 1, d_{w_2} = 2$$
$$w_3 : m_1 \succ_{w_3} m_2 \succ_{w_3} m_3, a_{w_3} = 2, d_{w_3} = 2$$

If the agents are truthful, the mechanism will match m_1 with w_1, m_2 with w_2 and m_3 with w_3. However, m_1 can report his preference as $w_2 \succ'_{m_1} w_1$. With this manipulation, he will get matched with w_2 in period 1, and remain available to match in period 2 with w_3, his most preferred woman. Thus, this greedy DA mechanism is manipulable.

3 Introducing a Fall-Back Option

A fall-back option allows a mechanism to decommit from a match made in an earlier period to a departed woman because the woman is assumed to have access to a fall-back option or substitute. Such a substitute is assumed to be at least as preferred as the match provided by the mechanism. On the other hand, substitutes are assumed to be costly to use and thus a woman would prefer to receive her match from the matching market.

Let R denote the set of substitutes. We now allow for a matching μ to allocate $\mu(m) \in W \cup \{\phi\}$ and $\mu(w) \in M \cup R \cup \{\phi\}$. For each substitute $r \in R$, we say that r is *equivalent* to man $m \in M$ for woman $w \in W$, if $m' \succ_w m \Leftrightarrow m' \succ_w r$ for all $m' \in M \setminus \{m\}$; i.e., as long as r is equivalent in terms of preference rank to m for woman w. In extending the notion of rank efficiency, the rank order to a woman for a substitute is that of the man m replaced by the substitute while the rank of the substitute himself is not included in *rank*(μ).

Definition 4. Matching μ is *stable* if there does not exist a blocking pair (m, w), where (m, w) is a blocking pair for μ if either:
(1) $w \succ_m \mu(m)$ and $m \succ_w \mu(w)$, or
(2) if w receives a substitute r that is equivalent to man m', then, $w \succ_m \mu(m)$ and $m \succ_w m'$.

When m is part of any blocking pair, we say m is *unstable*. Else we say m is *stable*.

3.1 GSODAS

Recall that $W(t)$ is the set of women present in period t. The GSODAS algorithm works as follows, where $\max_m(w_1, w_2)$ denotes the woman of $\{w_1, w_2\}$ most preferred by man m:

- For periods $t \in \{1, \ldots, T\}$, maintain *provisional match* $\mu^t(m) \in W \cup \{\phi\}$, for every $m \in M$. Initialize $\mu^0(m) = \phi$.
- Maintain a *committed* match $\mu^*(m)$ for every $m \in M$, initialized to $\mu^*(m) = \phi$ for all m.
- In every period t in which at least one woman departs,
  (i) run DA$(M, W(t))$, and let μ' denote this match
  (ii) update $\mu^t(m) := \max_m(\mu^{t-1}(m), \mu'(m))$ for every m
  (iii) if the assignment changes in $\mu^t(m)$ from $\mu^{t-1}(m)$ for man m, where $\mu^*(m) \neq \phi$ then woman $\mu^*(m)$ is matched with a substitute for m and $\mu^*(m) \leftarrow \phi$ with m no longer committed.
  (iv) $\mu^*(m) := \mu^t(m)$ if woman $\mu^t(m)$ departs in the current period.
- The final match μ_G has men matched as in $\mu^*(m)$ (along with corresponding $\mu(w)$ for matched women w), and with any other woman who received a substitute in step (iii) matched to this substitute, or otherwise unmatched.

GSODAS maintains a sequence of provisional matches μ^t in each period t, but that matches are only committed (and may even be subsequently decommitted) as women depart. A match is *valid* when no man is matched to multiple women and no woman is matched to multiple men.

Claim 3.1. *The GSODAS algorithm is strategyproof for men and generates a valid match.*

Proof. Fix man m. Strategyproofness follows immediately from the strategyproofness of man-proposing DA when one notices that the preferences reported by other agents in DA$(M, W(t))$ in period t are independent of the report of man $m \in M$. Moreover, man m receives the woman that is most preferred across all runs of the man-proposing DA, across all periods.

To establish that the final match is valid, suppose for contradiction that there is some woman $w = \mu_G(m_1) = \mu_G(m_2)$ for $m_1 \neq m_2$. Suppose that w is matched with m_1 at t_1 and m_2 at t_2. Assume first that $m_1 \succ_w m_2$. At t_2, w is matched with m_2, which implies that m_1 did not propose to her and received a better match at t_2 than w. But then we would not have $w = \mu_G(m_1)$ because this is the best match across all periods for m_1. Similarly, if $m_2 \succ_w m_1$ then at t_1, when w is matched with m_1, m_2 must have received a better match than w and we again have a contradiction. ☐

GSODAS matches every woman, either with a man $m \in M$ or with a substitute. Because $|M| = |W|$ and some women receive a substitute, the number of unmatched men equals the number of substitutes adopted in the mechanism.

3.2 Stability

Stability requires that there is no blocking pair, i.e., no man-woman pair that would both prefer to match with each other than their match from the mechanism.

Claim 3.2. *The GSODAS algorithm is stable.*

Proof. We prove the claim by a contradiction. Suppose a pair, (m, w) blocks the final match μ_G yielded by GSODAS. For each man, the final match is a woman most preferred among all his provisional matches (perhaps ϕ). Because $w \succ_m \mu_G(m)$, then m was never matched with w in a provisional match. Let a be the arrival time of w and d the departure time. Let $M(w) = \{m_a, m_{a+1}, \ldots, m_d\}$ denote the set of men with whom w is matched (if any) in the provisional match in each period $t \in \{a, \ldots, d\}$. Because the match generated in each period is stable, then $m' \succ_w m$ for all $m' \in M(w)$. In particular, we have $m_d \succ_w m$ and $\mu_G(w) \succ_w m$ (including the case where w later receives a substitute), and (m, w) is not a blocking pair. ☐

Claim 3.3. *The worst case substitute requirement in GSODAS for a T period problem, with $n = \alpha T$ men and women, for $\alpha \in \{1, 2, \ldots\}$, is $\alpha(T-1)$.*

m_1	$(T,\ldots,2,1,w)$	w_1	$(1,2,\ldots,T,m),\ a=d=1$
m_2	$(T,\ldots,2,1,w)$	w_2	$(\mu(w_1),1,2,\ldots,T,m),\ a=d=2$
$\vdots$			
m_T	$(T,\ldots,2,1,w)$	w_T	$(\mu(w_{T-1}),1,2,\ldots,T,m),\ a=d=T$
m_{T+1}	$(2T,\ldots,T+2,T+1,w)$	w_{T+1}	$(T+1,T+2,\ldots,2T,m),\ a=d=1$
m_{T+2}	$(2T,\ldots,T+2,T+1,w)$	w_{T+2}	$(\mu(w_{T+1}),T+1,\ldots,2T,m),\ a=d=2$
$\vdots$			
m_{2T}	$(2T,\ldots,T+2,T+1,w)$	w_{2T}	$(\mu(w_{2T-1}),T+1,\ldots,2T,m),\ a=d=T$
$\vdots$		$\vdots$	
$m_{(\alpha-1)T+1}$	$(\alpha T,\ldots,(\alpha-1)T+2,(\alpha-1)T+1,w)$	$w_{(\alpha-1)T+1}$	$(\mu(w_{(\alpha-1)T+1}),\ldots,\alpha T,m),\ a=d=1$
$\vdots$		$\vdots$	

Table 1. Construction of Agent Preferences Used for Worst-case Substitutes Requirement in Online Matching Mechanisms

Proof. Let k denote the number of matches between men and women, so that $n-k$ is the number of matches between women and substitutes. For k matches with (non-substitutes) men, we can have at most $(T-1)k$ substitutes, occurring when a better match is found for each of the k matched men in each round. We require k plus the total number of substitutes to be at least n, since all women will always receive some match. Therefore $k+(T-1)k \geq n$, and $k \geq n/T$. From this, the maximum number of substitutes, $n-k \leq n-n/T = \frac{(T-1)}{T}n = \alpha(T-1)$.

To see that this bound is tight, consider the following example. Consider an instance in which in every period exactly α women arrive and the j^{th} woman in that period indicates m_j to be her best match. Each woman departs immediately. That is, in period 1, women $w_1, w_2, \ldots, w_\alpha$ arrive and depart. In period i, $w_{(i-1)\alpha+1}, \ldots, w_{i\alpha}$ arrive and depart. $w_1, w_{\alpha+1}, w_{2\alpha+1}, \ldots, w_{(T-1)\alpha+1}$ indicate m_1 as the most preferred match. $w_2, w_{\alpha+2}, w_{2\alpha+2}, \ldots, w_{(T-1)\alpha+2}$ indicate m_2 as the most preferred match, and so forth. Each m_j has preference as $(w_{(T-1)\alpha+j}, \ldots, w_{\alpha+j}, w_j, w)$, where w is a placeholder for all other women (in arbitrary sequence). Each m_j, $j=1,2,\ldots,\alpha$ invokes the need for a substitute at every time $t=2,3,\ldots,T$, and therefore the total number of substitutes are $\alpha(T-1)$. $\square$

Thus, GSODAS has a large worst-case cost in terms of the number of substitutes required. We will evaluate an average-case cost in simulation. Comparing GSODAS with other algorithms, we establish a worst-case tradeoff between the number of substitutes and the number of men that can be part of a blocking pair. For this, define for matching μ the quantity,

$$S(\mu) = \big|\text{unstable men in } \mu\big| \;+\; \big|\text{substitutes used}\big|,$$

where an unstable man is part of at least one blocking pair.

Proposition 3.1. *For any online matching algorithm, for every problem with T periods, $n = \alpha T$ men and women, and $\alpha \in \{1,2,\ldots\}$, there exists an instance in which $S(\mu) \geq \alpha(T-1)$. For GSODAS, we have $S(\mu) \leq \alpha(T-1)$, with $S(\mu) = \alpha(T-1)$ in the worst case.*

Proof. Consider agent preferences in Table 1. The preference profile of a man m is denoted by the indices of women in decreasing order of preference; e.g., preference profile $w_2 \succ_m w_4 \succ_m w_1 \succ w$ will be denoted as $(2,4,1,w)$. The w at the end of the list indicates all other women in some arbitrary order. A similar convention is adopted for the preferences of women. The agents are grouped into α blocks, each consisting of T men and women. In each period, one woman from each block arrives and departs immediately. The groups are defined so that the men in each group prefer the women in the same group more than any woman in any other group. The same is true for the women, except that for any woman, w_{iT+j} for $i \in \{0,\ldots,\alpha-1\}$ and $j \in \{2,\ldots,T\}$, her most-preferred man is set to be the match $\mu(w_{iT+j-1})$ to the preceding woman in the block when this woman receives a match, and this match is not a substitute.

We argue that each of w_{iT+j} in groups $i \in \{0,\ldots,\alpha-1\}$ for $j \in \{1,\ldots,T-1\}$ contributes a count of 1 to $S(\mu)$. If such a woman receives a substitute, then she contributes 1 to this sum. Similarly, for every woman unmatched, at least one additional man is unmatched and part of a blocking pair (e.g., with the unmatched woman.) Now suppose that w_{iT+j} is matched with man $m_{i'T+j'}$ where $i' \neq i$. There must be some $w_{i'T+k}$ for $k \in \{1,\ldots,T\}$ not matched with a man in the i'th group. But then $(m_{i'T+j'}, w_{i'T+k})$ is a blocking pair because the man prefers any woman in i' to woman w_{iT+j} and woman $w_{i'T+k}$ prefers a man in group i' over a match from any other group, noting that for $k>1$ she cannot be matched to her most-preferred man for $\mu(w_{i'T+k-1}) \neq \phi$ because this man is matched with the preceding woman in the group. In the other case, when $i'=i$, then $(m_{i'T+j'}, w_{iT+j+1})$ is a blocking pair. This is because every man in group i prefers a later woman in the group over an earlier woman, and woman w_{iT+j+1} has $m_{i'T+j'}$ as her most-preferred match. Noting that for each such woman, w_{iT+j}, the blocking pair involves the man with whom she is matched, then we add 1 to $S(\mu)$.

In GSODAS, the number of unstable men $= 0$. And by Claim 3.2 the number of substitutes $\leq \alpha(T-1)$ and hence for GSODAS, $S(\mu) \leq \alpha(T-1)$. $\square$

We see that there is a tradeoff, in the worst-case, between the stability of an online algorithm and the number of substitutes. There exist instances where every substitute below $\alpha(T-1)$ leads to one additional man part of a blocking pair. For stability, then in the worst-case there is a need for at least as many substitutes as in GSODAS. An online algorithm that does not use substitutes will, in the worst-case, have a shrinking fraction $\alpha/n = 1/T$ of men that are *not* part of blocking pairs as T increases.

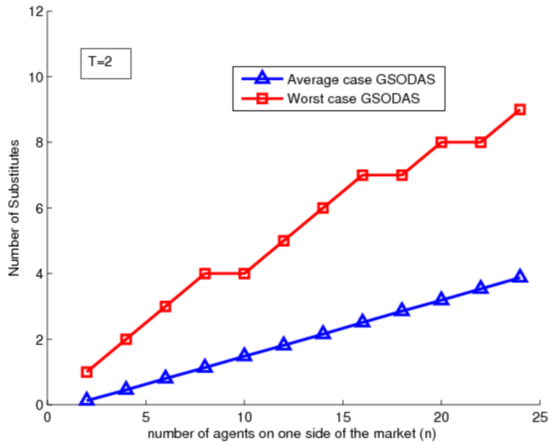

Figure 1. The number of substitutes required for men in GSODAS as n increases, fixing $T = 2$.

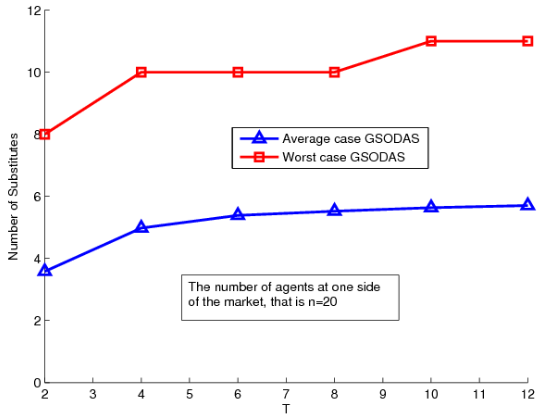

Figure 2. The number of substitutes required for men in GSODAS as T increases, fixing $n = 20$.

3.3 Randomized Online Matchings

In this section, we introduce two additional mechanisms, that are truthful for men but without using the fall-back option. These are Random Online Matching Algorithms (ROMA). In the first variation, ROMA1, every woman is matched with some man from the set M while in the second variation, ROMA2, not all the women are matched. The algorithms make different trade-offs between stability and rank-efficiency.

For ROMA1, in every period t, if there are departing women then select $|DW(t)|$ men at random and run man-proposing DA using these men and $DW(t)$. Commit to this match. In periods without departing women, then with probability $p > 0$ run man-proposing DA with $W(t)$ women and $|W(t)|$ men selected at random. Commit to this match. Any match is final and these men and women are not considered for matching in future periods. For ROMA2, we define a threshold $\tau \geq 1$, and whenever the number of women present is $|W(t)| \geq \tau$ then select $|W(t)|$ men at random and run man-proposing DA. Commit to this match.

Claim 3.4. *ROMA1 and ROMA2 are strategyproof for men.*

Proof. Men are randomly matched into a single instance of the man-proposing DA algorithm and cannot affect which instance they match to through misreports of preferences, and because the man-proposing DA is strategyproof for men. □

ROMA1 and ROMA2 have an advantage over GSODAS in that they do not require the use of substitutes. On the other hand, they may well lead a lot of blocking pairs and worse rank-efficiency because each man only participates in a single instance of DA.

3.4 Stochastic Optimization

To obtain a baseline performance for rank-efficiency we adopt an online sample-based stochastic optimization algorithm, based on the Consensus approach of Van Hentenryck and Bent [5]. The algorithm is not truthful, but provides good rank-efficiency.

The Consensus approach adopts a generative model of the future to sample random future arrivals of agents on the dynamic side of the market, and uses these samples to guide match decisions for agents in the market. In every period in which at least one woman departs, Consensus samples multiple possible future arrivals and matches each departing woman with the man with which she is most frequently matched when running a man-proposing DA on each sample:

For any period t in which at least one woman departs,

(i) generate K samples of the preferences for $n - \ell$ women, where ℓ women have already arrived,

(ii) for each sample W_k, for $k \in K$, run man-proposing DA$(M(t), W(t) \cup W_k)$

(iii) for each woman $w \in W(t)$, let $L(w)$ denote the man most frequently matched with her in the result of running DA on each of the K samples, breaking ties at random,

(iv) run man-proposing DA on the set of women, $W(t)$, and men in the set $\{L(w)|w \in W(t)\}$. Commit the matches in this DA that involve departing women, updating $M(t)$ accordingly.

Note that it is possible that $L(w_1) = L(w_2)$ for some $w_1 \neq w_2$, so that there are less men than women in step (iv) and some women may depart without a match.

4 Experimental Results

We compare the rank-efficiency and stability of GSODAS, ROMA1, ROMA2 and Consensus (which is not truthful). In all simulations, we generate preference profiles uniformly at random for all men and women. In ROMA1 the value of parameter p is set to be 0.3, which was found experimentally to provide good rank-efficiency for $T = 2$ and $T = 4$ for varying n. The threshold parameter τ in ROMA2 is similarly tuned to achieve the best performance for rank-efficiency, and we adopt $\tau = \max\{0.375\frac{n}{T}, 1\}$

We first investigate the number of substitutes required in GSODAS. For this we consider a problem with two time periods, increasing the number of agents on each side of the market from $n = 2$ to 24. For each woman i, a_i is either 1 or 2, both with equal probability and $d_i \in \{1, 2\}$ uniformly at random if $a_i = 1$, else $d_i = 2$. We also increase the number of periods T from 2 to 12, holding $n = 20$, and generating the arrival time, a_i, for a woman uniformly between $[1, T]$, with departure time d_i uniformly between $[a_i, a_i + T/3]$, with d_i also capped at a maximum value of T. In both experiments we determine worst-case and average case performance over

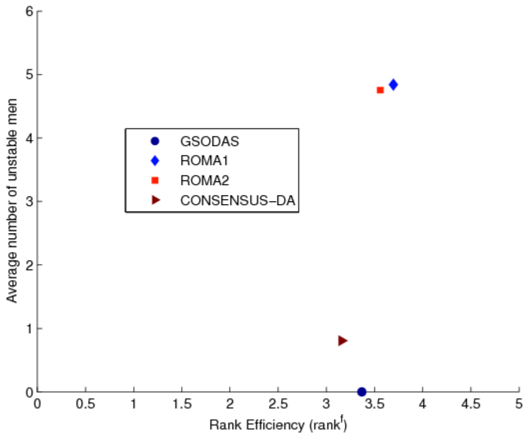

Figure 3. The rank-efficiency (x-axis) vs. the number of unstable men (y-axis) for $n = 10$ and $T = 2$.

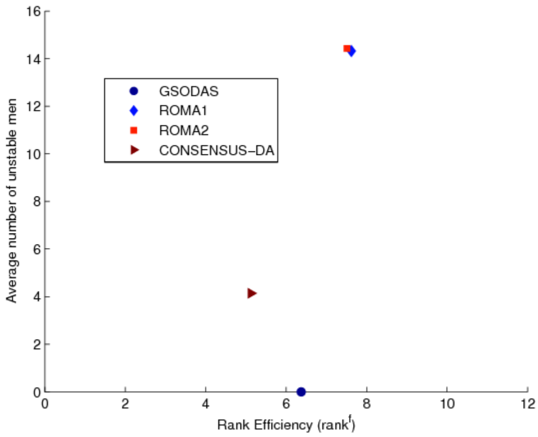

Figure 4. The rank-efficiency (x-axis) vs. the number of unstable men (y-axis) for $n = 20$ and $T = 4$.

20,000 random instances.

The results are illustrated in Figures 1 and 2. For a problem with two time periods, we find that an average of $\approx 20\%$ of the number of men are required as substitutes, increasing to around 30% for $T = 12$. For two period problems, in the worst case we need a substitute for as many as 1 in every 2 men in the market when $n \leq 10$; this fraction drops to 37% for $n = 24$. For $n = 20, T = 12$, then as many as 55% of the number of men are required as substitutes in the worst case.

We turn now to comparing rank-efficiency and stability in each of the mechanisms. For this, we determine the average rank-efficiency and average number of unstable men (i.e., number of men m for whom there exists a woman w such that (m, w) is a blocking pair). The results are again averaged over 20,000 instances. Figures 3 and 4 plot the average rank-efficiency (x-axis) against the average number of unstable men (y-axis) for $n = 10, T = 2$ and $n = 20, T = 4$, respectively. Recall that Consensus is not strategyproof, and that rank-efficiency assigns a rank of $(n + 1)$ to unmatched agents and ignores the rank preference of substitute agents. The results are encouraging for the GSODAS mechanism. We see that it dominates ROMA1 and ROMA2 in rank-efficiency while achieving perfect stability. This is even though we count $n + 1$ rank for the unmatched men in GSODAS, the number of which can be quite large due to to the use of substitutes. Comparing with Consensus, we see that GSODAS has worse rank-efficiency, achieving a rank-efficiency that is situated between that of Consensus and the ROMA mechanisms.

5 Conclusions

In this paper, we have initiated a study into dynamic matching problems in two-sided markets without money. One side of the market is static while the other side is dynamic, and we require truthfulness on the static side of the market. We achieve stability, and truthfulness on the static side, by allowing for the possibility of a fall-back option, so that the mechanism can decommit from some matches made to already departed agents, at which point a substitute is adopted. The GSODAS mechanism has better rank-efficiency than simpler methods that do not use substitutes, although with less rank-efficiency non-truthful stochastic optimization approach.

Still, the use of substitutes in GSODAS is quite high, with 30% on average as the number of agents and time periods increases (for uniform preferences) and as many as 55% required in the worst-case experimental instances. This is likely unacceptable in many practical domains, yet we prove that better worst-case properties are unavailable if full stability is required. The most interesting future direction, then, will look to relax the requirement of offline stability. This precludes blocking pairs, irrespective of the timing of the agents that comprise a blocking pair in system and the information available at the time of a match. Perhaps by relaxing this requirement, then mechanisms with good rank-efficiency, acceptable stability, but less need for exercising the fall-back option can be developed.

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