

Fair Package Assignment

Sébastien Lahaie* David C. Parkes†

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Abstract

We consider the problem of fair allocation in the package assignment model, where a set of indivisible items, held by single seller, must be efficiently allocated to agents with quasi-linear utilities. A fair assignment is one that is efficient and envy-free. We consider a model where bidders have superadditive valuations, meaning that items are pure complements. Our central result is that core outcomes are fair and even coalition-fair over this domain, while fair distributions may not even exist for general valuations. Of relevance to auction design, we also establish that the core is equivalent to the set of anonymous-price competitive equilibria, and that superadditive valuations are a maximal domain that guarantees the existence of anonymous-price competitive equilibrium. Our results are analogs of core equivalence results for linear prices in the standard assignment model, and for nonlinear, non-anonymous prices in the package assignment model with general valuations.

1. Introduction

In a package auction, agents place bids on bundles of items to account for the fact that items may be complements or substitutes. Interest in the design of package auctions has been driven by potential applications to problems such as the FCC’s allocation of wireless spectrum (Ausubel et al., 1997) and the FAA’s allocation of rights to landing slots (Ball et al., 2006). In the private sector, large-scale sealed-bid package auctions are already being run for procurement purposes (Sandholm, 2007).

The formal framework that underlies package auctions is known as the package assignment model (Bikhchandani and Ostroy, 2002), an extension of the standard assignment model (Shapley and Shubik, 1972). A single seller holds a set of distinct, indivisible items that must be allocated in a many-to-one fashion among agents with quasi-linear utilities, such that the total value of the allocation is maximized. The main questions addressed in this model so far concern market-clearing prices: whether they exist, and if so, whether they exhibit any special structure (Bikhchandani and Ostroy, 2002).

In this paper we consider the problem of fair allocation in the package assignment model. Since iterative package auctions are typically designed to reach a core outcome—in particular, a competitive equilibrium—we ask whether and to what extent core outcomes respect the classic fairness criteria of *no-envy* and *group no-envy*. We find that in the case of

*. Yahoo Research, New York, NY 10018.

`lahaies@yahoo-inc.com`

†. School of Engineering and Applied Sciences, Harvard University, Cambridge, MA 02138.

`parkes@eecs.harvard.edu`

general valuations, there may be no solutions, including the core, that satisfy these criteria. The situation is much better if we restrict our attention to superadditive valuations (i.e., pure complementarities). In that case core payoffs are always fair (efficient and envy-free) and even coalition-fair (efficient and group envy-free). Since the core is non-empty in our model, this proves the existence of fair and coalition-fair distributions of surplus.

There is a long literature on fairness in the standard assignment model with divisible items, originating with the no-envy concept of Foley (1967) and the work of Varian (1974). Of more direct relevance to our study is work concerning indivisible items (see among others Maskin, 1987; Alkan et al., 1991; Tadenuma and Thomson, 1991, 1993). Svensson (1983) in particular proves strong fairness properties of the core in the assignment model with indivisibilities: core outcomes are fair and even coalition-fair. As just mentioned, these results are mirrored here in the package assignment model with pure complementarities.

Our results are the first related to fairness properties of the core in the package assignment model. In the literature on fair allocation, one usually assumes that the agents have joint property rights over the items and money to be divided among them. In our model the property rights are held by a single seller, and the net amount of money available is zero—the payments issued by the agents are balanced by the revenue to the seller. Another distinguishing feature of our treatment is that we consider solutions as mappings from valuation profiles to distributions of surplus, rather than to outcomes (allocations and payments); this is consistent with the usual notion of a solution concept in cooperative game theory (see Osborne and Rubinstein, 1994). We say that a distribution of surplus is fair if every outcome from which the distribution arises is fair (efficient and envy-free). There may exist fair outcomes but no fair distributions.

Beviá (1998) studies package assignment from the perspective of joint property rights and examines the relationship between several fairness concepts such as envy and group no-envy, among others. She shows that with quasi-linear utilities, fair outcomes exist even with general valuations. This does not contradict our findings because of our stricter definition of fairness that applies to distributions of surplus rather than just outcomes. Our definition of a fair distribution of surplus does not depend on any tie-breaking rule used to select among efficient allocations.

Pápai (2003) also draws a close connection between fairness and superadditive valuations in the context of Groves mechanisms; these are efficient and truthful sealed-bid auctions that apply to package assignment. She finds that no Groves mechanism can ensure that the outcome is always fair given general valuations, but that many Groves mechanisms (which she characterizes) can guarantee fairness with superadditive valuations. Together with our results, this draws a close if non-obvious connection between pure complementarities and fairness. Our results complement hers because the core is disjoint from those distributions of surplus that arise from individually-rational Groves mechanisms.

After addressing the issue of fairness, we turn to the closely related question of *price discrimination* in the context of market-clearing. With general valuations, the seller must quote personalized prices (where different agents see different prices for the same bundle) in some instances to clear the market (Bikhchandani and Ostroy, 2002). Indeed some leading package auctions use price discrimination for this very reason (Ausubel and Milgrom, 2002; Parkes and Ungar, 2000). This clashes with our intuitive notion of “fairness” because the agents are not all offered the same deal. An anonymous-price competitive equilibrium is

envy-free by definition: no agent would want another agent’s bundle at the price the other agent pays, because both agents see the same prices.

With superadditive valuations, we argue that price discrimination is unnecessary. We show that anonymous competitive prices fill out the core over this domain, meaning that an anonymous price mechanism is in principle versatile enough to realize any core distribution of surplus. Existence of anonymous-price competitive equilibrium follows as an immediate corollary. Existence was first proved by Parkes and Ungar (2000), using the properties of their *i*Bundle auction. Our proof gives an explicit construction of anonymous competitive prices. We also provide a converse: existence of such prices no longer holds if we expand the domain by just a single subadditive valuation. This in fact holds for any sufficiently large domain that contains all single-minded valuations (whereby each bidder is only interested in acquiring a single, specific bundle).

Our focus on superadditive valuations is relevant because complements, rather than substitutes, are the central motivation for using package auctions. In the absence of complementarities, item-price auctions such as the simultaneous ascending auction can realize high levels of efficiency without the added complexity of package bidding (Cramton, 2006). With complementarities, an agent risks acquiring only a strict subset of the items it needs to derive nontrivial value. This *exposure problem* induces cautious bidding, leading to low efficiency and revenue. The problem is resolved if agents are allowed to bid on entire packages.

The remainder of the paper is organized as follows. Section 2 introduces the single-seller package assignment model together with the relevant valuation domains. Section 3 describes the fairness criteria that motivate normative solution concepts for our model, as well as the positive solution concepts of competitive equilibrium and the core. Section 4 shows that fair distributions and anonymous clearing prices may not exist with general valuations. Section 5 proves fairness properties of the core with superadditive valuations. In Section 6 we show that the core is equivalent to the set of anonymous-price competitive equilibria over the domain of superadditive valuations, and that the latter is a maximal domain that guarantees the existence of anonymous clearing prices. Section 7 describes extensions of our results to divisible items.

2. The Model

A seller wishes to allocate a set of indivisible items M among a set of agents $N = \{1, \dots, n\}$. We use the index 0 to refer to the seller. Let m be the number of items. An *allocation* is a vector of bundles $R = (R_i)_{i \in N}$, where R_j is the bundle allocated to agent j . An allocation is *feasible* if $R_i \cap R_j = \emptyset$ for $i \neq j$. We denote the set of feasible allocations by Γ . A *partition* is a set of bundles rather than a list of bundles, so that no assignment to the agents is defined.

An *outcome* is a feasible allocation R together with a vector of payments $q = (q_i)_{i \in N} \in \mathbf{R}^n$, denoted $\langle R, q \rangle$. Each agent i has a non-negative valuation function over bundles $v_i : 2^M \rightarrow \mathbf{R}_+$. The agents’ utilities over outcomes are quasi-linear, and there are no externalities, meaning that an agent only cares about the bundle it acquires and not what other agents obtain. Formally, agent i ’s utility for outcome $\langle R, q \rangle$ is of the form

$$u_i(R, q) = v_i(R_i) - q_i.$$

The fact that utilities are quasi-linear means that they can be denoted in a common currency, and utility can be transferred from the agents to the seller in the form of payments. In particular, agents are not constrained by any budget.¹ The fact that there are no externalities means that there can be interpersonal comparisons of the agents’ received bundles. This kind of comparison forms the basis of the fairness criteria to be introduced later. The seller has no value for any item or bundle of items; its utility is simply the revenue generated. Formally,

$$u_0(R, q) = \sum_{i \in N} q_i.$$

The seller may be viewed as an authority that coordinates the resource allocation process.² As such utility comparisons between agents and the seller are not used to define any fairness criteria later on.

An outcome is *efficient* if it maximizes the total utility to the agents and seller. We also say that an allocation is *efficient* if it is feasible and it maximizes the total value to the agents. Because utilities are quasi-linear, for any outcome $\langle R, q \rangle$ we have

$$\sum_{i \in N \cup \{0\}} u_i(R, q) = \sum_{i \in N} v_i(R_i),$$

so an outcome $\langle R, q \rangle$ is efficient if and only if R is efficient. The payments q only serve to redistribute surplus. We denote the set of efficient allocations by Γ^* .

A *coalition* is a subset of agents. Define the *coalitional value function* w over coalitions $L \subseteq N$ as

$$w(L) = \max_{R \in \Gamma} \sum_{i \in L} v_i(R_i).$$

This captures the maximum value that can be created by allocating items M solely among agents L , with the remaining receiving \emptyset . Here we have implicitly included the seller in the “coalition” along with the agents in L . In our model, coalitions that do not contain the seller create a total value of 0.

A *distribution (of surplus)* is a vector $\pi = (\pi_i)_{i=0}^n \in \mathbf{R}_+^{n+1}$. One typically assumes that the agents and seller have the option not to participate, in which case they pay and receive nothing. The lower bounds $\pi_i \geq 0$ for all $i \in N$ and $\pi_0 \geq 0$ for the seller are therefore natural in this context, which is why we restrict our attention to non-negative or *individually-rational* distributions.³ We say that an outcome $\langle R, q \rangle$ is *consistent* with a distribution π if $\pi_i = u_i(R, q)$ for all $i \in N$ and $\pi_0 = u_0(R, q)$. A distribution π is *efficient* if every outcome consistent with it is efficient. It is straightforward to check that this is equivalent to the condition that $\pi_0 + \sum_{i \in N} \pi_i = w(N)$.

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1. It is sufficient to assume that the budget of each agent i is at least $v_i(M)$, so quasi-linearity is a reasonable assumption when agents have ample liquidity or credit.
 2. A canonical example is the FCC’s role in allocating wireless spectrum.
 3. The term ‘distribution’ in this context follows Svensson (1983); it is short for ‘distribution of surplus’. There should be no confusion with the concept of a probability distribution because we make no use of probability in this work.

2.1 Valuations

A valuation is *monotone* if $S \subseteq T$ implies $v_i(S) \leq v_i(T)$. Monotonicity amounts to an assumption of free-disposal of the items. A valuation is *normalized* if the value for the empty set is $v_i(\emptyset) = 0$. The domain of *general valuations* consists of the monotone, normalized valuations. Throughout we only consider the domain of general valuations and subsets thereof.

The main results of this paper relate to pure complementarities. A valuation is super-additive if for any two bundles S, T such that $S \cap T = \emptyset$,

$$v_i(S) + v_i(T) \leq v_i(S \cup T).$$

A valuation is *subadditive* if the reverse inequality holds. Superadditivity captures the intuitive notion of complementarity: items are worth more together than separately. A valuation is single-minded if there is a bundle S such that

$$v_i(T) = \begin{cases} v_i(S) & \text{if } T \supseteq S \\ 0 & \text{otherwise} \end{cases}$$

A bidder with a single-minded valuation is interested in acquiring all the items in S , and no more. Single-minded valuations are superadditive; they are perhaps the simplest possible valuations exhibiting complementarity.

2.2 Prices

In traditional market models, prices are linear: a price is quoted for each item, and the price of a bundle is the sum of the prices of its constituent items. In the package assignment model, prices may be nonlinear and personalized. Under nonlinear prices, a distinct price may be quoted for each bundle. Under personalized prices, different agents may be quoted different prices for the same bundle, a practice known as price discrimination. Thus, in general, prices are of the form $p_i(S)$, for $i \in N$ and $S \subseteq M$. Prices are *anonymous* if $p_i(S) = p_j(S)$ for all $i \neq j$; in this case we will drop the agent subscript. Following Bikhchandani and Ostroy (2002), we identify three orders of pricing:

1. Linear and anonymous.
2. Nonlinear and anonymous.
3. Nonlinear and non-anonymous.

Third order prices have the same structure as valuation profiles, and the definitions of Section 2.1 apply just as well to nonlinear prices. We will speak of k -order prices, where $k \in \{1, 2, 3\}$, when our discussion or results apply to any order of pricing.

It is important to distinguish between payments and prices. Recall that, formally, payments are elements of \mathbf{R}^n . One payment is associated to each agent. A payment specifies what an agent is charged for the bundle it receives, but does not imply anything about what it would be charged for another bundle if it had the option to switch. Prices, on the other hand, are elements of $\mathbf{R}_+^{n2^m}$. They define a charge for every bundle and agent. Throughout we only consider prices that are non-negative, normalized, and monotone.

3. Solution Concepts

We first introduce the relevant normative solution concepts for our model, which prescribe outcomes that respect certain notions of fairness. We then describe the positive solution concepts used to specify the possible outcomes of a resource allocation process such as an auction. A central question addressed by our study is whether and to what extent the positive solutions respect the fairness criteria that motivate the normative solutions. The results in this section are standard or straightforward; proofs are provided in the appendix to make the presentation self-contained.

3.1 Normative Concepts

The criterion of *no envy* is one of the most studied in the literature on fair allocation. Following Varian (1974), we say that an agent *envies* another if it strictly prefers the other agent's bundle and payment to its own. Formally, given outcome $\langle R, q \rangle$, agent i envies agent j if

$$v_i(R_j) - q_j > v_i(R_i) - q_i.$$

An outcome is *envy-free* if no agent envies any other. An outcome is *fair* if it is efficient and envy-free.

We consider solution concepts that associate sets of distributions to valuation profiles, rather than sets of outcomes. Of course, an outcome implies a unique distribution of surplus. The reverse does not hold: several outcomes may be consistent with a distribution. Because efficient allocations are not unique in general, the distribution of surplus may vary depending on how ties are broken in a given mechanism. It stands to reason that a “fair” solution should not depend on a tie-breaking rule that favors any particular agent. Accordingly, we define fair distributions in such a way that they can arise whichever efficient allocation is selected: a distribution is *fair* if every outcome consistent with it is fair.

Note that if an efficient outcome $\langle R, q \rangle$ is consistent with distribution π , we have $\sum_{i \in N} \pi_i + \pi_0 = w(N)$. Thus if we take any other efficient allocation R' and define $q'_i = v_i(R'_i) - \pi_i$ for all $i \in N$, outcome $\langle R', q' \rangle$ is also consistent with π because $\sum_{i \in N} q'_i = w(N) - \sum_{i \in N} \pi_i = \pi_0$. Hence fair distributions are not tied to any particular efficient outcome. To characterize the set of fair distributions, let

$$\ell(i, j) = \min_{R \in \Gamma^*} v_i(R_i) - v_j(R_i)$$

for any two agents $i, j \in N$.

Lemma 1 *A distribution is fair if and only if it satisfies*

$$\pi_i - \pi_j \leq \ell(i, j)$$

for all $i, j \in N$, as well as $\sum_{i \in N} \pi_i + \pi_0 = w(N)$.

Beviá (1998) essentially defines the no-envy solution as the set of distributions with which *some* envy-free outcome is consistent. We can strengthen her solution concept and require consistency with some fair outcome (i.e., not just envy-free but also efficient). The resulting concept would still be more general than the fair solution (the set of fair distributions)

defined above, because a distribution is fair if every outcome consistent with it is fair, not just some outcome. Thus the fair solution may be empty even though there exist fair outcomes. Nevertheless, our fair solution generalizes the no-envy solution in the assignment problem, because there the no-envy solution coincides with the fair solution (Svensson, 1983). The positive concepts of competitive equilibrium and the core, defined below, are also independent of the choice of any particular efficient allocation.

An even stronger fairness criterion is that of *group no envy* (see Svensson, 1983; Beviá, 1998). Given an outcome, a coalition envies another of the same size if it can make each of its agents weakly better off, and one agent strictly better off, by acquiring the items of the other coalition and paying out the same total payment. Formally, given outcome $\langle R, q \rangle$ and coalitions $L, L' \subseteq N$ such that $|L| = |L'|$, coalition L' envies L if there is an outcome $\langle R', q' \rangle$ such that $\cup_{i \in L'} R'_i = \cup_{i \in L} R_i$, $\sum_{i \in L'} q'_i = \sum_{i \in L} q_i$, and

$$v_i(R'_i) - q'_i \geq v_i(R_i) - q_i$$

for all $i \in L'$, with strict inequality for some $j \in L'$. An outcome is *coalition-fair* if no coalition envies any other of the same size. A coalition-fair outcome is envy-free because coalitions of size one do not envy each other. A coalition-fair outcome is also efficient, because the coalition N does not envy itself, and this is equivalent to the efficiency of the outcome. We say that a distribution is *coalition-fair* if every outcome consistent with it is coalition-fair. In the assignment problem the fair and coalition-fair solutions coincide, and they are equivalent to the less restrictive group no-envy solution (Svensson, 1983).

3.2 Positive Concepts

When prices are used to coordinate resource allocation, as in an auction, *competitive equilibrium* is a natural positive solution. Indeed, iterative package auctions are usually designed so that they converge to a competitive equilibrium (for surveys see Cramton et al., 2006; de Vries and Vohra, 2003). Formally, a k -order competitive equilibrium $\langle R, p \rangle$, where $k \in \{1, 2, 3\}$, consists of a feasible allocation R together with k -order prices $p = (p_i)_{i \in N}$ such that for any other feasible allocation R' ,

$$v_i(R_i) - p_i(R_i) \geq v_i(R'_i) - p_i(R'_i) \quad \forall i \in N \quad (1)$$

$$\sum_{i \in N} p_i(R_i) \geq \sum_{i \in N} p_i(R'_i) \quad (2)$$

In a competitive equilibrium, each agent's allocated bundle maximizes the agent's utility at the given prices, and the chosen allocation also maximizes the seller's revenue at the given prices. In this sense, demand equals supply and the market clears. If $\langle R, p \rangle$ is a competitive equilibrium, we say that prices p *support* allocation R . The following standard result connects competitive equilibrium with the stated objective of efficiency.

Theorem 2 *If $\langle R, p \rangle$ is a competitive equilibrium, then R is efficient.*

We say that a competitive equilibrium $\langle R, p \rangle$ is consistent with a distribution π if the corresponding outcome $\langle R, q \rangle$, where $q_i = p_i(R_i)$ for all $i \in N$, is consistent with π . A distribution is *k -order competitive* if some k -order competitive equilibrium is consistent

with it. The following lemma implies that a competitive distribution is not tied to any particular choice of efficient allocation, consistent with our other solution concepts. The lemma applies to any order of prices.

Lemma 3 *If $\langle R, p \rangle$ is a competitive equilibrium consistent with distribution π , and R' is any efficient allocation, then $\langle R', p \rangle$ is a competitive equilibrium consistent with π .*

Note that neither Theorem 2 nor Lemma 3 assert the existence of a competitive equilibrium. The existence of competitive prices of a given order depends on the particular domain from which agent valuations are drawn.

There are other conceivable ways to distribute surplus other than price mechanisms. Our resource allocation problem can be viewed as a cooperative game with players $N \cup \{0\}$, where recall that 0 denotes the seller. The coalitional value function is clearly monotone and superadditive, because the agents' valuations are normalized, so $\langle N, w \rangle$ correctly defines a cooperative game (see, e.g., Osborne and Rubinstein, 1994). A central solution concept in cooperative game theory is *the core*. In our model, the *core* is the set of distributions π that satisfy

$$\pi_0 + \sum_{i \in N} \pi_i = w(N) \quad (3)$$

$$\pi_0 + \sum_{i \in L} \pi_i \geq w(L) \quad \forall L \subseteq N \quad (4)$$

$$\sum_{i \in L} \pi_i \geq 0 \quad \forall L \subseteq N \quad (5)$$

The core captures those distributions that are “stable” to deviations from coalitions. If an outcome is consistent with a distribution that does not lie in the core, then some coalition of agents, together with the seller, would have an incentive to reject it and instead realize a better outcome for itself. A coalition consisting solely of buyers can of course create a value of zero, so this is a lower bound for the total payoff of such coalitions, hence inequalities (5).

From this definition we can derive simple but useful lower and upper bounds on core distributions. Taking (5) with $L = \{i\}$, we have $\pi_i \geq 0$ for all $i \in N$, and taking (4) with $L = \emptyset$ we have $\pi_0 \geq 0$. Hence core distributions are individually-rational. Subtracting (4) with $L = N - i$ from (3), we obtain

$$\pi_i \leq w(N) - w(N - i) \quad (6)$$

for all $i \in N$. (Here $N - i$ is shorthand for $N \setminus \{i\}$.) This is the well-known fact that an agent's core payoff is upper-bounded by the agent's marginal contribution to the total value. The core is always non-empty in our model: the distribution with $\pi_0 = w(N)$ and $\pi_i = 0$ for $i \in N$ satisfies all the constraints. This is the “seller-optimal” core distribution at which the seller extracts all the surplus. By condition (3), core distributions are efficient, just like competitive distributions (of any order).

Besides selecting a core outcome, an often-suggested approach for package auctions is to run the Vickrey-Clarke-Groves (VCG) mechanism, which gives each agent i a payoff of $w(N) - w(N - i)$. The advantage of this is that agents are incentivized to report their true values. More generally, one could use a Groves mechanism to ensure truthfulness,

of which the VCG mechanism is a special case (see Vickrey, 1961; Clarke, 1971; Groves, 1979). Among individually-rational Groves mechanisms, the VCG mechanism maximizes the payment of each agent (Krishna and Perry, 2000). Therefore $\pi_i \geq w(N) - w(N - i)$ under any such mechanism. Comparing with (6), we see that distributions that arise from individually-rational Groves mechanisms lie outside the core, except perhaps for the one corresponding to the VCG outcome. The latter lies in the core only if an “agents are substitutes” condition holds, which is unlikely given superadditive valuations (Bikhchandani and Ostroy, 2002). Pápai (2003) studies fairness in Groves mechanisms. Because we focus on the core throughout, our analysis complements hers.

The concept of the core remains agnostic as to how players coordinate to actually realize the distribution of surplus. In this sense, the core is a more general concept than competitive equilibrium, which posits coordination by prices. This is formalized in the following standard result.

Theorem 4 *Competitive distributions lie in the core.*

Since core distributions are efficient, this can be seen as a strengthening of Theorem 2. The converse does not hold in general: core distributions are not necessarily competitive. If this is the case for a certain order of prices $k = 1, 2, \text{ or } 3$, then the sets of k -order competitive and core distributions are equivalent.

Core equivalence is appealing because it establishes that competitive equilibrium is unbiased: no coalition of agents can object to the use of a price mechanism for coordination on grounds that it would bias the outcome towards certain distributions of surplus within the core. Also, Ausubel and Milgrom (2002) have argued that core outcomes are particularly appealing in the context of package auctions because they guarantee a better revenue standard than the VCG outcome. Furthermore, a bidder-optimal core distribution minimizes the agents’ incentives to misreport their true values (Parkes et al., 2001; Day and Milgrom, 2007). Core equivalence ensures that any core distribution, in particular a bidder-optimal one, can in principle be reached by an appropriately designed package auction.

4. Impossibility of Fairness with General Valuations

We first show that with general valuations, there may be no fair distribution, and that price discrimination (third order prices) may be needed to clear the market.⁴ Consider the following example with two agents and three items.

Example 1 *The set of agents is $N = \{1, 2\}$ and the set of items is $M = \{a, b, c\}$. The agents’ valuations are given in the following table.*

S	a	b	c	ab	ac	bc	abc
$v_1(S)$	5	2	5	6	6	7	7
$v_2(S)$	2	5	5	6	7	6	7

4. Bikhchandani and Ostroy (2002) also provide an example for the latter, drawn from Kelso and Crawford (1982). They use linear programming arguments whereas we show non-existence directly from the definition of competitive equilibrium.

There are several possible efficient allocations, such as (ac, b) and (a, bc) . By Lemma 1 a fair distribution π must satisfy both

$$\begin{aligned}\pi_1 - \pi_2 &\leq v_1(ac) - v_2(ac) = -1 \\ \pi_2 - \pi_1 &\leq v_1(bc) - v_2(bc) = -1\end{aligned}$$

which is impossible. Because the valuations are subadditive,⁵ we conclude that fair and coalition-fair solutions may not exist over the domains of subadditive or general valuations. In contrast, recall that the core is always non-empty in our model.

The same example demonstrates that third order prices may be needed to clear the market. For the sake of contradiction, let p be second order prices that support efficient allocation (a, bc) . Since the agents' bundles maximize their utilities at these prices, we have

$$\begin{aligned}5 - p(a) &\geq 7 - p(bc) \\ 6 - p(bc) &\geq 5 - p(b) \\ 6 - p(bc) &\geq 7 - p(ac)\end{aligned}$$

Summing these inequalities and rearranging, we find that

$$p(b) + p(ac) > p(a) + p(bc),$$

which contradicts the fact that allocation (a, bc) should maximize the seller's revenue at prices p . Hence second order competitive prices do not exist. For the domains of subadditive or general valuations, third order prices are needed to guarantee the existence of a competitive equilibrium.

The example above is as simple as possible. With just two items, there is always a fair distribution over the domain of general valuations, and second order competitive prices always exist. (The proof of this is left to the reader.)

5. Fairness of the Core with Complementarities

In this section we show that with superadditive valuations, fair and coalition-fair distributions exist. In contrast with the assignment model, however, the two solutions do not necessarily coincide. We will in fact show that the core is a strict subset of the coalition-fair solution, which immediately implies that the latter is non-empty since the core is always non-empty in our model.

As a first step, we show that the core distributions are fair.

Lemma 5 *Let R be an efficient allocation, and let π be in the core. If the agents have superadditive valuations, then*

$$v_i(R_i) - \pi_i \geq v_j(R_j) - \pi_j$$

for all $i, j \in N$.

5. In fact, the valuations are submodular: $v_i(S) + v_i(T) \geq v_i(S \cup T) + v_i(S \cap T)$ for all $S, T \subseteq M$.

Proof Consider the allocation where agent i obtains \emptyset , j obtains $R_j \cup R_i$, and every other agent k receives R_k as before. Note that

$$\begin{aligned} w(N-i) &\geq v_j(R_i \cup R_j) + \sum_{k \neq i, j} v_k(R_k) \\ &\geq v_j(R_i) + v_j(R_j) + \sum_{k \neq i, j} v_k(R_k) \end{aligned} \quad (7)$$

where the first inequality follows from the definition of w , and the second from the fact that v_j is superadditive. We then have

$$\begin{aligned} \pi_i &\leq w(N) - w(N-i) \\ &\leq v_i(R_i) - v_j(R_i) \end{aligned}$$

where the second inequality follows from (7) above and the fact that $w(N) = \sum_{k \in N} v_k(R_k)$. Hence $v_i(R_i) - \pi_i \geq v_j(R_i)$, and since $\pi_j \geq 0$, it follows that $v_i(R_i) - \pi_i \geq v_j(R_i) - \pi_j$. ■

Because the efficient allocation in the statement of Lemma 5 is arbitrary, the lemma implies that core distributions satisfy the characterization given in Lemma 1. As the core is non-empty, we immediately obtain the following.

Corollary 6 *There always exists a fair distribution over the domain of superadditive valuations.*

We now turn to our first main result. It can be proved from first principles, but it is quicker and more intuitive to appeal to the characterization of the core given later in Theorem 9, which appeals to Lemma 5.

Theorem 7 *The core is coalition-fair over the domain of superadditive valuations.*

Proof Let π be a distribution in the core. Let $\langle R, q \rangle$ be an outcome consistent with this distribution; R is necessarily an efficient allocation. Assume for the sake of contradiction that there are coalitions $L, L' \subseteq N$ such that $|L| = |L'|$ and L' envies L . Specifically, let $\langle R', q' \rangle$ be an outcome such that

$$\bigcup_{i \in L'} R'_i = \bigcup_{i \in L} R_i \quad (8)$$

$$\sum_{i \in L'} q'_i = \sum_{i \in L} q_i \quad (9)$$

and for all $i \in L'$,

$$v_i(R'_i) - q'_i \geq v_i(R_i) - q_i \quad (10)$$

with strict inequality for some $j \in L'$. Let $K \subseteq M$ be the set of items allocated to L under R (equivalently, to L' under R').

Summing (10) over all $i \in L'$, and taking into account that at least one inequality is strict, we obtain

$$\sum_{i \in L'} [v_i(R'_i) - q'_i] > \sum_{i \in L'} [v_i(R_i) - q_i]. \quad (11)$$

By Theorem 9, there are second order prices p that support allocation R such that $\langle R, p \rangle$ is consistent with π ; in particular, $\pi_i = v_i(R_i) - p(R_i)$ and therefore $p(R_i) = q_i$ for all $i \in N$. By the definition of competitive prices we have $v_i(R_i) - p(R_i) \geq v_i(R'_i) - p(R'_i)$ for all $i \in N$, and summing these over all $i \in L'$ we obtain

$$\sum_{i \in L'} [v_i(R_i) - p(R_i)] \geq \sum_{i \in L'} [v_i(R'_i) - p(R'_i)]. \quad (12)$$

Combining (11) with (12) and rearranging, we obtain

$$\sum_{i \in L'} p(R'_i) > \sum_{i \in L'} q'_i. \quad (13)$$

By the definition of competitive prices, allocation R maximizes revenue at prices p . Because the prices are anonymous, $\{R_i\}_{i \in L}$ must be a revenue-maximizing partition of the items K into $|L|$ bundles. Note that by (8), $\{R'_i\}_{i \in L'}$ is also a partition of K into $|L'| = |L|$ bundles. Therefore,

$$\sum_{i \in L} p(R_i) \geq \sum_{i \in L'} p(R'_i). \quad (14)$$

From (13), (14), and the fact that $p(R_i) = q_i$, we obtain

$$\sum_{i \in L} q_i > \sum_{i \in L'} q'_i,$$

which contradicts (9) and completes the proof. ■

Because the core is non-empty in our model, we immediately obtain the following.

Corollary 8 *There always exists a coalition-fair distribution over the domain of superadditive valuations.*

The next example shows that coalition-fair distributions are not necessarily in the core with single-minded (and hence superadditive) valuations, so the core is a more stringent solution concept. A slight change to the example shows that fair distributions are not necessarily coalition-fair over the single-minded domain.

Example 2 *The set of agents is $N = \{1, 2, 3\}$ and the set of items is $M = \{a, b\}$. The agents' valuations are single-minded and given in the following table.*

S	a	b	ab
$v_1(S)$	2	0	2
$v_2(S)$	0	2	2
$v_3(S)$	0	0	2

Consider the efficient outcome with $R = (a, b, \emptyset)$ and $q = (0, 0, 0)$, and let π be the implied distribution. The latter is not in the core because $w(\{3\}) = 2$ while $\pi_0 + \pi_3 = 0$. It is straightforward to check that no agent envies any other agent. The grand coalition does not envy itself because the allocation is efficient. We show that coalition $\{1, 3\}$ does not

envy $\{1, 2\}$; the other pairs of size two can be checked by a similar argument. Coalition $\{1, 2\}$ obtains bundle ab for a total payment of 0. We will try to allocate ab to $\{1, 3\}$ and charge payments q'_1, q'_3 such that $q'_1 + q'_3 = 0$, in such a way that each agent weakly prefers the result to $\langle R, q \rangle$, with one agent strictly preferring the result.

Consider first allocating (\emptyset, ab) to agents 1 and 3, respectively. Payment q'_1 and q'_3 must be such that $v_1(\emptyset) - q'_1 \geq 2$ and $v_3(ab) - q'_3 \geq 0$, with one of these strict. This reduces to $q'_1 \leq -2$ and $q'_3 \leq 2$, with one of these strict, so we must have $q'_1 + q'_3 < 0$. This conflicts with the requirement that $q'_1 + q'_3 = 0$. It is simple to check that the same kind of conflict arises with allocations (ab, \emptyset) , (a, b) , and (b, a) for 1 and 3 respectively. We have established that outcome $\langle R, q \rangle$ consistent with π is coalition-fair. This is the only outcome consistent with π because (a, b, \emptyset) is the unique efficient allocation. Therefore π is coalition-fair but does not lie in the core.

Now suppose we change the example slightly such that $v_3(ab) = 3$ and all other values remain the same. Again consider the outcome with $R = (a, b, \emptyset)$ and $q = (0, 0, 0)$. The implied distribution π is fair because: (i) it is simple to check that $\langle R, q \rangle$ is fair, and (ii) there are no other fair outcomes consistent with π since R is still the unique efficient allocation. However, $\langle R, q \rangle$ is not coalition-fair because $\{1, 3\}$ envies $\{1, 2\}$. To see this, we can give \emptyset to agent 1 and ab to agent 3, and charge payments $q'_1 = -2$ and $q'_3 = 2$. Agent 1 weakly prefers this outcome to the original, while agent 2 strictly prefers it.

With two items and less than three agents, fair distributions are coalition-fair (trivially), and the core is equivalent to the fair solution. With just one item the three concepts coincide.

6. Equivalence of Anonymous Competitive Prices and the Core

In this section we examine the relationship between the competitive and core solutions. As explained in Section 3.2, the competitive solution of any order is always contained in the core. We show that second order competitive distributions fill out the core over the domain of superadditive valuations. This implies that second order competitive prices always exist. In fact, we show that the superadditive valuations are a maximal domain with this property.

For general valuations, Bikhchandani and Ostroy (2002) showed that if π is a core distribution, then the third order prices

$$p_i(S) = \max\{v_i(S) - \pi_i, 0\} \tag{15}$$

support any efficient allocation R , and $\langle R, p \rangle$ is consistent with π . We will show that if the valuations are superadditive, the same is achieved by the anonymous prices

$$p(S) = \max_{i \in N} \max\{v_i(S) - \pi_i, 0\}. \tag{16}$$

It is simple to check that both prices (15) and (16) are monotone and normalized. However, prices (16) are not necessarily superadditive, even if the valuations are superadditive.

The intuition behind the following proof is clearest when one keeps in mind the seller-optimal distribution for concreteness: $\pi_0 = w(N)$ and $\pi_i = 0$ for all $i \in N$.

Theorem 9 *The second order competitive solution is equivalent to the core over the domain of superadditive valuations.*

Proof Let R be an efficient allocation, and let π be in the core. Note that $w(N) = \sum_{i \in N} v_i(R_i)$. Let prices p be defined by (16). For each $i \in N$, we have

$$\begin{aligned} v_i(R_i) &= \sum_{j \in N} v_j(R_j) - \sum_{j \neq i} v_j(R_j) \\ &\geq w(N) - w(N - i) \\ &\geq \pi_i, \end{aligned}$$

where the first inequality follows from the definition of w . Hence $v_i(R_i) - \pi_i \geq 0$ for all i . Because R is efficient and π is in the core, it then follows from Lemma 5 that $p(R_i) = v_i(R_i) - \pi_i$. Hence $v_i(R_i) - p(R_i) = \pi_i$, and for all $S \subseteq M$, $v_i(S) - p(S) \leq v_i(S) - [v_i(S) - \pi_i] = \pi_i$ by the definition of p . Therefore, for each $i \in N$, R_i maximizes agent i 's utility at prices p , and the agent obtains a payoff of π_i . Also note that

$$\begin{aligned} \sum_{i \in N} p(R_i) &= \sum_{i \in N} v_i(R_i) - \sum_{i \in N} \pi_i \\ &= w(N) - \sum_{i \in N} \pi_i \\ &= \pi_0, \end{aligned}$$

so the seller receives payoff π_0 .

We next show that R maximizes the seller's revenue at prices p . Among revenue-maximizing allocations, choose an allocation R' that maximizes the number of agents that receive nothing. We claim that we cannot have two nonempty bundles R'_i and R'_j such that $p(R'_i) = v_k(R'_i) - \pi_k$ and $p(R'_j) = v_k(R'_j) - \pi_k$; that is, the prices of both bundles cannot be derived from the same agent's valuation and core payoff. If this were the case, we would have

$$\begin{aligned} p(R'_i) + p(R'_j) &= v_k(R'_i) + v_k(R'_j) - 2\pi_k \\ &\leq v_k(R'_i \cup R'_j) - \pi_k \\ &\leq p(R'_i \cup R'_j) + p(\emptyset) \end{aligned}$$

by the superadditivity of v_k and the definition of p , and the fact that $\pi_k \geq 0$. We see then that replacing R'_i with $R'_i \cup R'_j$ and R'_j with \emptyset would result in an allocation with weakly greater revenue, but with one more agent receiving nothing, which would contradict our original choice of R' .

If $R'_i \neq \emptyset$ and $p(R'_i) > 0$, reassign the bundle so that $p(R'_i) = v_i(R'_i) - \pi_i$. By our arguments above, this is a well-defined reassignment of such bundles. The remaining bundles with $p(R'_i) = 0$ can be reassigned to the remaining agents arbitrarily. Because prices are anonymous, this reassignment does not change the revenue, and so we can assume without loss of generality that this was the original assignment. Let N' be the agents that receive

<i>Domain</i>	<i>CE = Core</i>	<i>Reference</i>
Additive	1	(immediate)
Unit-Demand, Substitutes	3	Kelso and Crawford (1982)
Single-minded, Superadditive	2	this work
Subadditive, General	3	Bikhchandani and Ostroy (2002)

Table 1: The lowest order of prices such that the core is equivalent to competitive equilibrium (CE), for various common domains.

a bundle with positive price under allocation R' . The revenue from R' is then

$$\begin{aligned}
\sum_{i \in N} p(R'_i) &= \sum_{i \in N'} [v_i(R'_i) - \pi_i] \\
&\leq w(N') - \sum_{i \in N'} \pi_i \\
&\leq \pi_0 + \sum_{i \in N'} \pi_i - \sum_{i \in N'} \pi_i \\
&= \sum_{i \in N} p(R_i).
\end{aligned}$$

where the second inequality follows from the fact that π is in the core. Since R' was revenue-maximizing at prices p , so is R .

The constructed prices ensure that each bundle in R maximizes its respective agent's utility, and that the allocation maximizes revenue to the seller. Therefore they are second order competitive prices that support R , such that $\langle R, p \rangle$ is consistent with π . ■

Bikhchandani and Ostroy (2002) provide a core equivalence result for third order prices over the domain of general valuations, and Theorem 9 applies to second order prices over the domain of superadditive valuations. First order competitive distributions fill out the core when agents have additive valuations. This is not the case if the agents have unit-demand valuations, as in the standard assignment model (Shapley and Shubik, 1972), or substitutes valuations, in the sense of Kelso and Crawford (1982), because the seller cannot extract all the surplus.⁶ These conclusions are collected in Table 1.

As the core is non-empty in our model, Theorem 9 immediately implies the following.

Corollary 10 *There exist second order competitive prices over the domain of superadditive valuations.*

This result was first shown by Parkes and Ungar (2000), who provide an algorithmic proof based on the properties of their *i*Bundle auction. Our approach provides an explicit construction of second-order competitive prices according to (16).

6. This may seem to contradict the results of Shapley and Shubik (1972). The reason is that the pattern of ownership in their model is different from ours. Each item is owned by a distinct seller, and hence no seller can extract all the surplus.

Bikhchandani and Ostroy (2002) showed that third order competitive prices exist given general valuations. For valuations that satisfy the “substitutes” condition, Kelso and Crawford (1982) showed that first order competitive prices exist.⁷ Corollary 10 is an analog of these results for second order pricing.

The following result provides a converse to Corollary 10. There may be specific valuation profiles with subadditivities that still admit second order competitive prices. However, if the domain is “sufficiently large” (it contains all single-minded valuations) and just a single valuation with subadditivity is introduced, existence can no longer be guaranteed.

Theorem 11 *Suppose that the domain from which agent valuations are drawn contains the domain of single-minded valuations, and that $n \geq 3$. Then second order competitive prices exist for every possible profile of valuations only if every valuation in the domain is superadditive.*

Proof Assume there is a valuation v_1 in the domain with strict subadditivity: there exist nonempty $S, S' \subseteq M$ such that $S \cap S' = \emptyset$ and $v_1(S) + v_1(S') > v_1(S \cup S')$. Let v_2 be a single-minded valuation with $v_2(T) = v_1(S \cup S')$ for $T \supseteq S \cup S'$ and $v_2(T) = 0$ otherwise. Finally, let v_3 be a single-minded valuation with $v_3(T) = v_1(M) + v_2(M)$ for $T \supseteq M \setminus (S \cup S')$ and $v_3(T) = 0$ otherwise. The valuations of any remaining agents are set to 0 over all bundles.

If agent 3 is not given a superset of $M \setminus (S \cup S')$, no value greater than $v_1(M) + v_2(M)$ can be achieved, so there exists an efficient allocation where agent 3 gets such a superset. On the other hand, giving the agent more than items $M \setminus (S \cup S')$ cannot add any value, so there is an efficient allocation where it receives exactly these items. To allocate the remaining items $S \cup S'$ efficiently, note that agent 2 only gets positive value if it obtains all these items, so it is efficient to either give the agent either all these items or none of them. We see then that allocation $(\emptyset, S \cup S', M \setminus (S \cup S'))$ is efficient.

Assume there exist normalized, monotone, and anonymous prices p that support this allocation. Because \emptyset maximizes agent 1’s utility at these prices, we have

$$v_1(S) - p(S) \leq 0 \tag{17}$$

$$v_1(S') - p(S') \leq 0 \tag{18}$$

$$v_1(S \cup S') - p(S \cup S') \leq 0 \tag{19}$$

Agent 2 must prefer its bundle $S \cup S'$ to the empty set, so we also have

$$v_2(S \cup S') - p(S \cup S') \geq 0. \tag{20}$$

From (19) and (20) we see that $p(S \cup S') = v_1(S \cup S') = v_2(S \cup S')$. We then have

$$\begin{aligned} p(S) + p(S') &\geq v_1(S) + v_1(S') \\ &> v_1(S \cup S') \\ &= p(S \cup S'), \end{aligned}$$

7. In words, the substitutes condition says that if, under linear prices, the price of an item is increased, an agent’s demand for the other items does not decrease. This captures a certain notion of substitutability, because the agent substitutes away from the item that sees a price increase.

<i>Domain</i>	<i>CE exists</i>	<i>Reference</i>
Additive	1	(immediate)
Unit-Demand	1	Koopmans and Beckmann (1957)
Substitutes	1	Kelso and Crawford (1982)
Single-minded, Superadditive	2	this work
Subadditive, General	3	Bikhchandani and Ostroy (2002)

Table 2: The lowest order of prices that guarantees that a competitive equilibrium (CE) exists, for various common domains.

where the first inequality follows from (17) and (18), and the second by assumption. But this means that the revenue from allocation $(S, S', M \setminus (S \cup S'))$ is strictly greater than the revenue from $(\emptyset, S \cup S', M \setminus (S \cup S'))$; so prices p cannot in fact support the latter, which gives us a contradiction. ■

Theorem 11 is analogous to a result of Ausubel and Milgrom (2002) for substitutes valuations. They show that the class of substitutes valuations is a maximal class containing the additive valuations that guarantees the existence of first order CE prices. Similarly, general valuations are (trivially) a maximal class for which the existence of third order competitive prices is guaranteed. Theorem 11 provides an analog of these facts for superadditive valuations and second order prices.

The following example shows that first order competitive prices do not necessarily exist when agents have single-minded and hence superadditive valuations.

Example 3 *The set of agents is $N = \{1, 2, 3, 4\}$ and the set of items is $M = \{a, b, c\}$. The agents' valuations are single-minded. Agent 1 wants abc and values it at 4. Agent 2 wants ab and values it at 3. Agent 3 wants bc and values it at 3. Agent 4 wants ac and values it at 3.*

It is efficient to give abc to agent 1 and nothing to the others. Assume there exist first order competitive price that support this allocation. Since agents 2, 3, and 4 receive nothing, we must have

$$\begin{aligned} p(a) + p(b) &\geq 3 \\ p(b) + p(c) &\geq 3 \\ p(a) + p(c) &\geq 3 \end{aligned}$$

from which it follows that

$$p(a) + p(b) + p(c) \geq 9/2.$$

But in order for abc to maximize agent 1's utility, its price must be below 4. So we have reached a contradiction, and first order competitive prices cannot exist.

The example is as simple as possible: if there are three items and less than four agents, or four agents and less than three items, then it can be shown that first order competitive prices exist with single-minded valuations. (The proof of this is left to the reader.) The conclusions on the existence of various orders of competitive prices are collected in Table 2.

7. Extensions

We have so far considered a model with unit supply of indivisible items. This was for notational and conceptual simplicity, and to be consistent with previous studies, in particular those related to the assignment problem. Our results in fact extend to cases where there is an arbitrary finite supply of divisible or indivisible items.

For each $j \in M$, let $X_j \subseteq \mathbf{R}_+$ be a closed set such that: (i) X_j is closed under addition, and (ii) $0 \in X_j$. The set X_j can be construed as the possible quantities of j that can be consumed. The first condition ensures that quantities can be combined, and the second that it is always possible to consume nothing. Now let $X = \times_{j \in M} X_j$ be the consumption set common to all agents. A “bundle” is now an element of X . The consumption set is non-empty because it contains $\mathbf{0}$, the empty bundle.

Let $z = (z_j)_{j \in M} \in X$ be the seller’s endowment. Two bundles s and t are disjoint if we have $s + t \leq z$. The definitions of normalized, monotone, superadditive and single-minded valuations are simple to adapt. For instance, a valuation v_i is superadditive if $v_i(s + t) \geq v_i(s) + v_i(t)$ for disjoint bundles s and t . The set of feasible allocations is now $\Gamma = \{x \in X^n : \sum_{i \in N} x_i \leq z\}$. By our assumptions on X , it is compact (closed and bounded). Thus if we insist that each agent have a *continuous* valuation over X , there exists an efficient allocation. If we further insist that prices be continuous, then there always exists a utility-maximizing bundle for each agent, and there exists a revenue-maximizing allocation. Note that the constructions (15) and (16) are continuous given continuous valuations.

The specific instance of this general model that was considered in this paper had $X_j = \mathbf{Z}_+$ and $z_j = 1$ for all $j \in M$. It is possible to consider divisible items as well, where $X_j = \mathbf{R}_+$, or combinations of divisible and indivisible items. The general model also allows for arbitrary finite supply.

It is not difficult to verify that under this more general model, all the results in this paper carry through (except of course for the counterexamples, which are constructed specifically for the case of unit supply and indivisible items). For instance, consider the simple model with just one unit of a single divisible item, where $X = X_1 = \mathbf{R}_+$, and $z = z_1 = 1$. Suppose each valuation v_i exhibits increasing marginal values. Such valuations are superadditive; therefore taking their upper envelope according to (16) yields second-order competitive prices.

8. Conclusions

While the received literature has focused on substitutes valuations (Ausubel and Milgrom, 2002; Bikhchandani and Ostroy, 2002), we argue that complements are a central motivation for package auctions because they are the source of the exposure problem. We found that under pure complementarities, core distributions exhibit strong fairness properties. This connection between complementarities, fairness, and the core is all the more relevant given that package auctions find important applications in the public sector; this includes auctions for wireless spectrum, airport take-off and landing slots, electricity, and bus routes, among others (see Cramton et al., 2006).

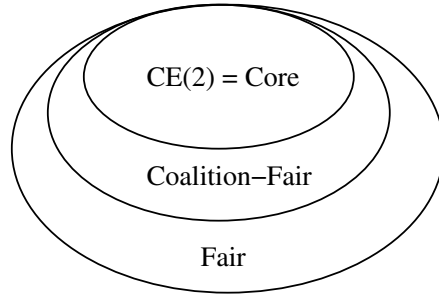


Figure 1: Inclusion relationships between the fair, coalition-fair, second-order competitive and core solutions over the domain of superadditive valuations.

We showed that over the domain of superadditive valuations, the core is equivalent to the set of anonymous-price competitive equilibria, and that core distributions are fair and even coalition-fair. The relationships between these different solution concepts are depicted in Figure 1. With subadditive or general valuations, core distributions are not necessarily fair; in fact, a fair distribution may not even exist whereas the core is always non-empty.

Our core equivalence result is of relevance to auction design for several reasons. It demonstrates that under pure complementarities, there is no need to resort to price discrimination to achieve certain core distributions of surplus. In particular, bidder-optimal core distributions can be realized in anonymous-price competitive equilibrium. These have emerged as appealing solutions for package auctions because they ensure that revenue is always monotonically increasing in the number of agents, and that revenue always dominates that of the VCG mechanism, among other nice properties (Ausubel and Milgrom, 2002; Day and Milgrom, 2007). Price discrimination can also be problematic because it presupposes restrictions on resale that might be costly or impossible to enforce in certain settings (Bikhchandani and Ostroy, 2002).

A typical approach used to develop iterative package auctions is to formulate the allocation problem as a linear program, and then interpret dual methods (e.g., primal-dual or subgradient) on this program as auctions (de Vries et al., 2007). Our core equivalence result implies that the anonymous-price linear programming formulation given by Bikhchandani and Ostroy (2002) in fact characterizes the core with superadditive valuations. This opens up the possibility of developing dual methods on this program (i.e., anonymous-price auctions) that reach whichever core outcome is most appropriate for a given package assignment problem.

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Appendix: Proofs for Section 3

Proof of Lemma 1. Let π be a distribution satisfying the conditions of the lemma. Let $\langle R, q \rangle$ be an outcome consistent with π . The total value of R is $\sum_{i \in N} v_i(R_i) = \sum_{i \in N} \pi_i + \pi_0 = w(N)$, so it is efficient. For each $i \in N$ and $j \neq i$, we have

$$\begin{aligned} \pi_j - \pi_i &\leq \ell(j, i) \\ \pi_j - \pi_i &\leq v_j(R_j) - v_i(R_i) \\ v_i(R_i) - [v_i(R_i) - \pi_i] &\geq v_i(R_j) - [v_j(R_j) - \pi_j] \\ v_i(R_i) - q_i &\geq v_i(R_j) - q_j \end{aligned}$$

Therefore $\langle R, q \rangle$ is envy-free as well, so it is fair. As $\langle R, q \rangle$ was arbitrary, π is fair. Reversing the argument shows that the conditions are sufficient for fairness. ■

Proof of Theorem 2. Let $\langle R, p \rangle$ be a competitive equilibrium. Given a feasible allocation R' , summing inequalities (1) and (2) yields $\sum_{i \in N} v_i(R_i) \geq \sum_{i \in N} v_i(R'_i)$. Since R' was arbitrary, R is efficient. ■

Proof of Lemma 3. Let $\langle R, p \rangle$ be a competitive equilibrium consistent with π , and let R' be an efficient allocation. We have

$$\begin{aligned} \sum_{i \in N} v_i(R'_i) &= \sum_{i \in N} [v_i(R'_i) - p_i(R'_i)] + \sum_{i \in N} p_i(R'_i) \\ &\leq \sum_{i \in N} [v_i(R_i) - p_i(R_i)] + \sum_{i \in N} p_i(R_i) \\ &= \sum_{i \in N} v_i(R_i). \end{aligned} \tag{21}$$

But because R' is efficient, $\sum_{i \in N} v_i(R_i) \leq \sum_{i \in N} v_i(R'_i)$, and so $\sum_{i \in N} v_i(R'_i) = \sum_{i \in N} v_i(R_i)$. Hence inequality (21) holds with equality, and each R'_i maximizes i 's utility at prices p_i , while R' maximizes revenue at prices p . This shows that $\langle R', p \rangle$ is a competitive equilibrium, such that for all $i \in N$

$$v_i(R_i) - p_i(R_i) = v_i(R'_i) - p_i(R'_i) = \pi_i,$$

as well as

$$\sum_{i \in N} p_i(R'_i) = \sum_{i \in N} p_i(R_i) = \pi_0.$$

Therefore $\langle R', p \rangle$ is consistent with π . ■

Proof of Theorem 4. Let $\langle R, p \rangle$ be a competitive equilibrium. Let $\pi_i = v_i(R_i) - p_i(R_i)$ and $\pi_0 = \sum_{i \in N} p_i(R_i)$ be the corresponding competitive payoffs. For any coalition of agents L , let R' be an efficient allocation of the items among them (where $R'_i = \emptyset$ for $i \notin L$). Then,

$$w(L) = \sum_{i \in L} v_i(R'_i) = \sum_{i \in L} [v_i(R'_i) - p_i(R'_i)] + \sum_{i \in N} p_i(R'_i) \leq \sum_{i \in L} \pi_i + \pi_0, \tag{22}$$

where the inequality follows from (1) and (2). We have

$$\sum_{i \in N} \pi_i + \pi_0 = \sum_{i \in N} v_i(R_i) \leq w(N),$$

and combined with (22) for $L = N$, the inequality holds with equality. Hence π is in the core. ■