On Indirect and Direct Implementations of Core Outcomes in Combinatorial Auctions

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Abstract

This note presents a partial characterization of the core outcome implemented in the ascending-proxy combinatorial auction, which is demonstrated to terminate at a core point intermediate between buyer-optimal core and buyer-optimal recursive-core solutions. In addition, we propose a simple variation to ascending-proxy that always implements a buyer-optimal core outcome and implements the VCG payoffs whenever they are in the core. This retains the useful robustness to shill bids and joint deviations, but removes the bargaining problem when agents-are-substitutes but the stronger buyer-submodular condition fails. In lieu of a complete characterization of the outcome implemented by the ascending-proxy auction we do introduce a semi-direct implementation that runs the auction as a sequence of stages and may prove particularly useful in proxy settings with rich bidder-proxy preference languages. Finally, we present a number of direct implementations of buyer-optimal core outcomes, and hope to start a debate about appropriate selection criteria. This is important in environments in which agents fail to solve the bargaining problem, which is implicit in the core outcome when VCG payoffs are outside the core, amongst themselves.

1 Introduction

One agenda item that arose from the recent FCC spectrum-auction meeting was to investigate direct-implementations of the ascending-proxy auction. This is interesting for two reasons. First, identifying the outcome in the core that is implemented by the ascending-proxy auction, with respect to reported agent valuations, will help to open up a discussion about the appropriate selection criteria to address the implicit bargaining problem that exists between agents when one implements core outcomes and VCG payments are outside the core. Second, it would be nice to provide a rich bidder-proxy interface, for example to allow bidders to express valuations by constructing "business scenarios". However, there are concerns that it may be unreasonable computationally to implement combinatorial auction outcomes, based on these rich preference structures, within the current XOR-language based ascending-price auction framework. A direct characterization of the outcome implemented by the ascending-proxy auction would lead immediately to a direct implementation.

In outline, Section 2 introduces notation, and defines important concepts. Section 3 introduces examples that help to illustrate problems and observations brought out in the discussion. Section 4 develops our understanding of the core outcomes that are selected by the ascending-proxy auction, and suggests a simple variation to implement a buyer-optimal core outcome. We also review a simple extension that provides an indirect implementation of the VCG outcome, and also helps to characterize the outcome of ascending-proxy when agents-are-substitutes. Section 5 suggests a number of selection criteria to select buyer-optimal core outcomes in direct-revelation mechanisms, and also introduces a semi-direct implementation of the ascending-proxy auction. Section 6 closes with some open problems. The Appendix includes a few technical details about some of the algorithms touched on the discussion.

2 Preliminaries

Agents $\mathcal{I} = \{0, 1, \ldots, N\}$, indexed $i \in \mathcal{I}$, that includes the seller, 0. Discrete items, $\mathcal{G}$, and bundles, $S \subseteq \mathcal{G}$ of items. Agent valuations, $v_i(S) \geq 0$, for $S \subseteq \mathcal{G}$, and free-disposal with $v_i(S) \geq v_i(T)$ for all $S \supseteq T$. The seller is assumed to have $v_0(S) = 0$ for all bundles S. A feasible allocation, $S = (S_1, \ldots, S_N)$, allocates bundle S_i to agent i, and requires $S_i \cap S_j = \emptyset$ for all $i \neq j$. Let $\mathcal{X}$ denote the space of feasible allocations.

Definition 1 (coalitional value function). *Let $w(K) = \max_{S \in \mathcal{X}} \sum_{i \in \mathcal{K} \setminus 0} v_i(S_i)$, for all coalitions K that include the seller, and $w(K) = 0$ otherwise.*

It is convenient to use CAP(K) to denote the efficient allocation problem restricted to agents $K \subseteq \mathcal{I}$. We also write S^* to denote the efficient allocation with the full set of agents.

Let $\pi = (\pi_0, \ldots, \pi_N)$ denote a vector of agent payoffs. Given allocation, $S = (S_1, \ldots, S_N)$, payments $p = (p_1, \ldots, p_N)$, and budget-balance, then payoffs π are computed as $\pi_0 = \sum_{i \in \mathcal{I} \setminus 0} p_i$ and $\pi_i = v_i(S_i) - p_i$ for all $i \neq 0$. By budget-balance, we have $\sum_{i \in \mathcal{I}} \pi_i = \sum_{i \in \mathcal{I} \setminus 0} p_i + \sum_{i \in \mathcal{I} \setminus 0} (v_i(S_i) - p_i) = w(S)$.

Definition 2 (core payoffs). *The core for a coalition, K, is defined as the set of agent payoffs that are feasible and unblocked by a coalition $L \subset K$:*

$$Core(K, w) = \{\pi \ : \ w(K) = \sum_{i \in K} \pi_i, \ w(L) \leq \sum_{i \in L} \pi_i \ \forall L \subset K\} \tag{CORE}$$

where the condition $w(K) \leq \sum_{i \in K} \pi_i$ is combined with feasibility to give $w(K) = \sum_{i \in K} \pi_i$.

Unless otherwise noted, *core payoffs* simply refer to payoffs that are in the core for the full set, $\mathcal{I}$, of agents. Consider an allocation, S. Payoffs that correspond to that allocation, for some balanced payment scheme, can only be in the core if S is efficient. In addition, core payoffs must satisfy, $\pi_i \leq v_i(S_i^*)$, for all $i \in \mathcal{I} \setminus 0$, where S^* is the efficient allocation. To see this, suppose otherwise, that $\pi_j > v_j(S_j^*)$ for some $j \neq 0$. Now, $\pi_j = w(\mathcal{I}) - (\pi_0 + \sum_{i \in \mathcal{I} \setminus \{0,j\}} \pi_i > v_j(S_i^*)$, which implies that $\sum_{i \in \mathcal{I} \setminus j} \pi_i \leq w(\mathcal{I}) - v_j(S_j^*) \leq w(\mathcal{I} \setminus j)$, and that the payoffs to coalition $\mathcal{I} \setminus j$ are blocked.

Equivalently, payoffs in the core can be expressed as $(w(\mathcal{I}) - \sum_{i \in \mathcal{I} \setminus 0} \pi_i, \pi_1, \ldots, \pi_N)$, with constraints:

$$w(\mathcal{I}) - w(\mathcal{I} \setminus K) \geq \sum_{i \in K} \pi_i, \quad \forall K \subset \mathcal{I}, 0 \ \notin K \tag{ALT-CORE}$$

We introduce a stronger solution concept, the *recursive-core*, which ties in with the condition of buyer-submodular values. Let π_L denote the vector of payoffs imputed on agents in set $L \subset \mathcal{I}$, i.e. $\pi_{1,2} = (\pi_1, \pi_2)$.

Definition 3 (recursive core). *The recursive-core is defined as the set of agent payoffs,* $\pi \in RecCore(\mathcal{I}, w)$*, that are in* $Core(\mathcal{I}, w)$*, and also impute payoffs* $(w(L) - \sum_{i \in L \setminus 0} \pi_i, \pi_L)$ *that are in the core for every subcoalition, such that:*

$$w(L) - w(L \setminus K) \geq \sum_{i \in K} \pi_i, \quad \forall K \subset L, 0 \notin K \qquad \text{(REC-CORE)}$$

for every $L \subset \mathcal{I}$*,* $0 \in L$*.*

In other words, payoffs π_{-0}, are in the core, if the payoffs are also in the core (with values for π_0 selected to give feasibility) for every subcoalition, $L \subset \mathcal{I}$.

We also introduce an intermediate solution concept, the *universal core*[1], that allows the implementation of VCG payoffs in an indirect mechanism.

Definition 4 (universal core). *The universal-core is defined as the set of agent payoffs,* $\pi \in UnivCore(\mathcal{I}, w)$*, that are in* $Core(\mathcal{I}, w)$*, and also impute payoffs,* $(w(L) - \sum_{i \in L \setminus 0} \pi_i, \pi_L)$ *that are in the core for every subcoalition,* $L = \mathcal{I} \setminus j$*, for some* $j \neq 0$*.*

In other words, the universal-core is intermediate between the core and the recursive-core, requiring that payoffs satisfy core constraints for $\mathcal{I}$ and for all $\mathcal{I} \setminus j$ in which one agent $j \neq 0$ is removed.

There is a direct equivalence between core payoffs and competitive equilibrium outcomes. To see this, let $p_i(S) \geq 0$, for $S \subseteq \mathcal{G}$, denote prices on bundles. We allow these prices to be both non-linear and non-anonymous in general. In addition, assume that prices are consistent with free-disposal, such that $p_i(S) \geq p_i(T)$ for all $S \supset T$, and $p_i(\emptyset) = 0$ for all i.

Definition 5 (competitive equilibrium). *An allocation,* S*, and prices,* p*, are in competitive equilibrium if and only if the following two conditions hold:*

$$v_i(S_i) - p_i(S) = \max_{S' \subseteq \mathcal{G}} (v_i(S') - p_i(S')), \quad \forall i \in \mathcal{I} \setminus 0 \qquad \text{(CS1)}$$

$$\sum_{i \in \mathcal{I} \setminus 0} p_i(S_i) = \max_{S' \in \mathcal{X}} \sum_{i \in \mathcal{I} \setminus 0} p_i(S'_i) \qquad \text{(CS2)}$$

where $\emptyset \subseteq \mathcal{G}$ *by definition.*

In words, (S, p) is a competitive equilibrium if the allocation S maximizes the payoff for all agents, including the seller, at the prices. It is an immediate consequence of linear-programming (LP) duality in application to appropriate LP formulations of the combinatorial allocation problem [BO02] that all CE outcomes are efficient.

Lemma 1 (core and CE equivalence). *Payoffs are in the core in every competitive equilibrium and every set of core payoffs is supported in some competitive equilibrium.*

Proof. See the appendix. □

This equivalence serves to unify the presentations and analysis of *i*Bundle [Par99, PU00a] with the ascending-proxy auction [AM02].[2] In all that follows "ascending-proxy auction" can be replaced with "*i*Bundle(3) with a direct-revelation proxy-agent interface".

[1] This terminology follows the description of *universal competitive equilibrium prices* in Parkes & Ungar [PU02].

[2] First, a core outcome can be supported with linear prices when preferences satisfy the *substitutes* property [KC82, GS00], and this is almost a necessary condition [Mil00]; we have a partial characterization of when a core outcome can be supported with non-linear but anonymous prices [PU00a]; and all core outcomes can be supported in some competitive equilibrium (perhaps non-linear and non-anonymous). Second, *i*Bundle [Par99, PU00a] implements an outcome in the core with myopic best-response strategies. Third, a proxy-agent direct-revelation mechanism (DRM) implementation of *i*Bundle implements an outcome in the core for *revealed* valuations, and all core outcomes can be implemented in a Nash equilibrium with semi-sincere bidding strategies.

Definition 6 (VCG payoffs). *The payoffs,* π_{vcg}*, are computed as*

$$\pi_{\text{vcg},i} = w(\mathcal{I}) - w(\mathcal{I} \setminus i), \quad \forall i \in \mathcal{I} \setminus 0 \tag{VCG}$$

and $\pi_{\text{vcg},0} = w(\mathcal{I}) - \sum_{i \in \mathcal{I} \setminus 0} \pi_{\text{vcg},i}$.

Buyer-optimal core payoffs provide a useful correspondence with VCG payoffs. Let $\pi^B \subseteq Core(\mathcal{I}, w)$ denote the set of buyer Pareto-optimal core payoffs. Let $\overline{\pi}(i) = \max_{\pi \in \pi^B} \pi_i$ denote agent i's most-preferred buyer-optimal core payoff. Agent i's payoff at core outcome, $\overline{\pi}(i)$, equals its VCG payoff [PU00b, AM02]. In the language of competitive equilibrium, the buyer-optimal core payoffs are supported in *group-minimal* CE prices and agent i's VCG payoff is supported in the *individual-minimal* CE prices for agent i [PU02].

VCG payoffs are simultaneously supported in the core if and only if a technical condition, *agents-are-substitutes*, holds on preferences [Aus97, BO02]. This is immediate, given the alternative definition (ALT-CORE) of the core conditions.

Definition 7 (agents-are-substitutes). *A condition over the coalitional value function that requires:*

$$w(\mathcal{I}) - w(\mathcal{I} \setminus K) \geq \sum_{i \in K} \left[w(\mathcal{I}) - w(\mathcal{I} \setminus i) \right], \quad \forall K \subset \mathcal{I},\ 0 \notin K \tag{AAS}$$

In fact, VCG payoffs are supported in the core exactly when there is a unique buyer-optimal core payoff vector [PU00b, AM02].

A stronger condition, *buyer-submodular* values, is sufficient for the ascending-proxy auction [AM02] to terminate with VCG payoffs.

Definition 8 (buyer-submodular). *[AM02] A condition over the coalitional value function that requires:*

$$w(L) - w(L \setminus K) \geq \sum_{i \in K} \left[w(L) - w(L \setminus i) \right], \quad \forall K \subset L,\ 0 \notin K \tag{BSM}$$

for all $L \subseteq \mathcal{I}$*,* $0 \in L$.

Buyer-submodular requires that the Vickrey payoff vector is in the core for all coalitions, L, not just with all agents.[3] It is immediate from the definition of the recursive core (REC-CORE) that VCG payoffs are in the recursive-core if and only if buyer-submodularity holds.

3 Illustrative Examples

The discussion of indirect and direct implementations of core and VCG outcomes that follows is illustrated with respect to the following examples. In describing the outcome of the ascending proxy auction we assume *myopic best-response* (MBR), which corresponds to an agent that provides its true valuation to the proxy.

[3]Goods are substitutes [KC82] is sufficient, and almost necessary for buyer-submodularity [AM02]. This is quite a pessimistic result, because it shows that bargaining is a problem in the ascending-proxy and *i*Bundle auctions when goods are not substitutes, because there are already quite effective auctions for the substitutes case [GS00].

example 1. [AM02, p.28] $v_1(AB) = 10, v_2(CD) = 20, v_3(CD) = 25, v_4(BD) = 10, v_5(CA) = 10$, $\pi_{\text{vcg}} = (20, 10, 0, 5, 0, 0)$, and core payoffs requires $\{\pi_1 \leq 10, \pi_3 \leq 5, \pi_1 + \pi_3 \leq 15\}$. AAS holds and the VCG payoffs are in the core. However BSM fails, with AAS violated for coalition $L = \{0, 1, 4, 5\}$.[4] The ascending-proxy auction terminates with core payoffs $(30, 0, 0, 5, 0, 0)$.

example 2. $v_1(AB) = 10, v_2(CD) = 20, v_3(CD) = 25, v_4(BD) = 8, v_5(CA) = 10$, $\pi_{\text{vcg}} = (20, 10, 0, 5, 0, 0)$, and core requires $\{\pi_1 \leq 10, \pi_3 \leq 5, \pi_1 + \pi_3 \leq 15\}$. AAS holds and the VCG payoffs are in the core. However BSM fails, with AAS violated for $L = \{0, 1, 4, 5\}$. The ascending-proxy auction terminates with core payoffs $(29, 1, 0, 5, 0, 0)$.

example 3. $v_1(A) = v_1(B) = 8, v_2(A) = v_2(B) = 8, v_3(AB) = 10$, $\pi_{\text{vcg}} = (4, 6, 6, 0)$, and core payoffs require $\{\pi_1 \leq 6, \pi_2 \leq 6, \pi_1 + \pi_2 \leq 6\}$. AAS fails, and the VCG payoffs are not in the core, with π_{vcg} blocked by coalition {0,3}. The ascending-proxy auction terminates with core payoffs, $(10, 3, 3, 0)$.

example 4. $v_1(A) = v_1(B) = 16, v_2(A) = v_2(B) = 8, v_3(AB) = 10$, $\pi_{\text{vcg}} = (2, 14, 8, 0)$, and core payoffs require $\{\pi_1 \leq 14, \pi_2 \leq 8, \pi_1 + \pi_2 \leq 14\}$. AAS fails, and the VCG payoffs are not in the core. The ascending-proxy auction terminates with core payoffs, $(10, 11, 3, 0)$.

Examples 1 and 2 show that the ascending-proxy auction does not always terminate with a buyer-optimal core outcome.

4 Indirect Implementations

In this section, we develop our understanding of the core outcomes that are selected by the ascending-proxy auction. We are not able to provide a simple and direct formulation of the core point selected by the auction, but gain enough intuition to develop a "semi-direct" implementation which is outlined in the next section. We also observe that a very simple variation on the ascending-proxy auction will always implement a buyer-optimal core payoff outcome [PU00b, PU02]. This extension strengthens the theoretical properties of the ascending-proxy auction, bringing truth-revelation into equilibrium whenever the agents-are-substitutes condition holds, and without requiring the stronger buyer-submodular condition. It comes at no cost to the robustness of the ascending-proxy auction equilibrium to shill bids and joint deviations, which accrue from the bidder monotonicity property of the auction that follows from properties of the core.

4.1 Ascending-Proxy Auction

The problem addressed in this section is to characterize the core outcome implemented in the ascending-proxy auction *with respect to the valuations provided to the proxy agents*. We know that the ascending-proxy auction terminates in the core with respect to reported valuations [PU00a, AM02]. The analysis of Ausubel & Milgrom [AM02] provides an additional characterization for the following two special cases:

the core is singular The ascending-proxy auction terminates in the unique core point whenever the core with respect to reported valuations is singular. This follows immediately from the fact that the auction terminates in the core. This simple observation can be used to analyze the semi-sincere strategy Nash equilibrium of the auction, in which agents implement a buyer-optimal core point [AM02, theorem 4]. Suppose π' is a *bidder-optimal* point in the core.

[4] The VCG payoffs are $\pi'_{\text{vcg}} = (0, 0, 10, 10)$ for subproblem $L = \{0, 1, 4, 5\}$, which violate AAS.

Then semi-sincere reports, $\hat{v}_i(S) = \max(0, v_i(S) - \pi'_i)$, are a Nash equilibrium of the auction. The auction implements payoffs (with respect to reported valuations) of $\hat{\pi} = (\hat{w}(\mathcal{I}), 0, \ldots, 0)$, where $\hat{w}(\cdot)$ is the coalitional value function based on revealed preferences.[5] The *actual* payoffs, with respect to the true valuations of agents, are exactly the core payoffs, π'.[6]

buyer-submodular values Suppose the reports, $\hat{v}_i(S)$, satisfy buyer-submodularity. Then, the buyer-optimal core payoff with respect to these reported values is unique and equal to VCG payoffs, again for the reported values. Moreover, the auction implements this buyer-optimal core payoff vector [AM02, theorem 8].

The second condition usefully characterizes the equilibrium outcome of the auction when preferences satisfy buyer-submodularity. However, there remains a large gap in the understanding of the economic properties of the particular core point selected by the agents when agents fail to solve the bargaining problem that is implicit in the core outcome when agents-are-substitutes fails. The equilibrium analysis in Theorem 4 assumes that agents solve the bargaining problem amongst themselves and play the corresponding Nash equilibrium.

The following characterization follows from the analysis of Theorem 8 [AM02].

Proposition 1 (payoff dominance). *The payoffs implemented in the ascending-proxy auction, given reported valuations, $\hat{v}_i(\cdot)$, buyer Pareto-dominate the payoffs in all recursive-core outcomes.*

Proof. Consider a buyer-optimal recursive-core payoff vector, $\overline{\pi} \in RecCore(\mathcal{I}, w)$. Consider agent j, and suppose that there is some round, t, at which $\pi_j^t < \overline{\pi}_j$. We show that j is in the winning coalition in that round. Let K be any coalition including the seller but not agent j. Then,

$$\begin{aligned} w(K) - \sum_{i \in K} \pi_i^t &< w(K) - \sum_{i \in K} \pi_i^t + (\overline{\pi}_j - \pi_j^t) \\ &\leq w(K) - \sum_{i \in K \cup \{j\}} \pi_i^t + w(K \cup j) - w(K) \qquad (1) \\ &= w(K \cup j) - \sum_{i \in K \cup \{j\}} \pi_i^t \end{aligned}$$

where (1) follows from the recursive core constraints (REC-CORE). □

Theorem 8 [AM02] follows as a corollary, because with buyer-submodular values the buyer-optimal core payoffs are unique and equal to the buyer-optimal recursive-core payoffs.

We already know, from Examples 1 and 2, that the ascending-proxy auction does not implement a buyer-optimal core outcome. Moreover, the examples show that the auction does not even implement the buyer-optimal core outcome when agents-are-substitutes and there is no bargaining problem, but requires the stronger condition of buyer-submodular values. We also observe, from Examples 2, 3 and 4, that the auction does not implement a buyer-optimal recursive core outcome,

[5] To see that this is the only core solution to the coalitional game over reported values, let $\mathcal{I}^* \subseteq \mathcal{I} \setminus 0$ denote the *winners*, the agents that receive a non-empty bundle in the efficient allocation. At reported valuations, $\hat{v}_i(\cdot)$, the coalitional value, $\hat{w}(\mathcal{I}) = \hat{w}(\mathcal{I}(\mathcal{I}^*)$. Any core payoffs that are not blocked require $\sum_{i \in \mathcal{I} \setminus \mathcal{I}^*} \geq \hat{w}(\mathcal{I} \setminus \mathcal{I}^*)$, and since $\hat{\pi}_i = 0$ for all $i \notin \{\mathcal{I}^*, 0\}$, then $\hat{\pi}_0 \geq \hat{w}(\mathcal{I} \setminus \mathcal{I}^*)$. Feasibility then requires $\hat{\pi}_i = 0$ for all $i \in \mathcal{I}^*$, and the core payoffs are completely pinned down.

[6] The efficient allocation for the revealed valuations is unchanged, which gives $\pi_i = \pi'_i$ for all $i \neq 0$. Finally, $\pi_0 = \hat{w}(\mathcal{I}) = w(\mathcal{I}) - \sum_{i \in \mathcal{I} \setminus 0} \pi'_i = \pi'_0$.

although this is the case in Example 1. [7] The auction often terminates with payoffs that are outside the recursive-core, but that buyer-dominate all recursive-core payoffs, by Proposition 1.

In summary:

(1) the auction implements payoffs in the core, but these need not be buyer-optimal payoffs

(2) the payoffs weakly dominate all buyer-optimal recursive-core payoffs

(3) the payoffs can be neither buyer-optimal core not buyer-optimal recursive-core payoffs, but somewhere in between.

Loosely speaking, the auction implements an outcome "inbetween that of the buyer-optimal core and the buyer-optimal recursive-core payoffs". Recursive-core conditions are stronger than core conditions, and capture the intuition that subcoalitions of agents are in dynamic competition during the auction. It is the effects of this dynamic competition that can lead to payoffs that are not buyer-optimal core payoffs. In Example 1, the payoff to agent 1 is driven below its buyer-optimal core payoff because the agent must initially compete, in coalitions {1,2} and {1,3} with coalition {4,5}. The high valuations of agents 2 and 3 are not known in these early rounds, and agent 1 participates in the competitive bidding process. Only when coalition, {4,5}, drops out of the auction can agent 1's payoff stop falling, while agents 2 and 3 compete to form the winning coalition with 1.[8] One can imagine that agents in subproblem {0,1,4,5} compete against each other in Example 1, driving the payoff to agent 1 down to zero.

Continuing, we might consider the following conjecture: *the ascending-proxy auction implements a buyer-optimal outcome subject to core constraints and some* ***subset*** *of additional recursive-core constraints.* Perhaps the characterization problem reduces to determining *which* recursive-core constraints to include, or "which subcoalitions are in dynamic competition during the auction", in addition to a selection criteria to choose a particular buyer-optimal payoff vector and solve the implicit bargaining problem?

Although the following analysis shows that this approach does *not* lead to a full characterization of the outcome of the ascending-proxy auction, it does nevertheless yield some interesting insights. For example, an *equalize-payments* selection criteria, in combination with an appropriate selection of recursive-core constraints, is able to characterize the outcome of the auction in Examples 1, 3 and 4.

Definition 9 (equalize-payments criteria). *The objective in the equalize-payments selection criteria is to select buyer-optimal payoffs, π, to solve*

$$\min_{\pi \in \Pi} \max_{i \in \mathcal{I} \setminus 0} [v_i(S_i^*) - \pi_i] \qquad \text{(EQUAL-PAY)}$$

where Π is set of feasible payoffs, implied by the core and selected recursive-core constraints, and $S^* = (S_1^*, \ldots, S_N^*)$ is the efficient allocation.

[7] In Example 1 the VCG payoffs, $\pi_{\text{vcg}} = (20, 10, 0, 5, 0, 0)$ violated the recursive-core for coalitions $\{0, 1, 4, 5\}$, $\{0, 1, 4\}$ and $\{0, 1, 5\}$. For example, the payoffs in $L = \{0, 1, 4, 5\}$ induced by π_{vcg} are $\pi = (10, 10, 0, 0) \notin Core(\{0, 1, 4, 5\}, w)$, because they are blocked by {0,4,5}. However, the outcome of the auction, $\pi = (30, 0, 0, 5, 0, 0)$, satisfy the recursive core, which requires $\pi_1 \leq 0$ and $\pi_3 \leq 5$. In Example 2 the auction terminates with $\pi = (29, 1, 0, 5, 0, 0)$, and the payoffs induced in $L = \{0, 1, 4, 5\}$ are $\pi' = (17, 1, 0, 0)$, which are blocked by coalition $\{0, 4, 5\}$. Recursive core payoffs in Example 2 require conditions $\{\pi_1 \leq 0, \pi_3 \leq 5\}$. Similarly, in Example 3, the auction terminates with core payoffs $\pi = (10, 3, 3, 0)$, but recursive-core payoffs require $\{\pi_1 = \pi_2 = 0\}$. In Example 4, the auction terminates with payoffs, $\pi = (10, 11, 3, 0)$, but recursive core requires $\{\pi_1 \leq 6, \pi_2 \leq 0\}$.

[8] For example, if $v_1(AB) = 15$, then agent 1's final payment remains 10 and its payoff in the auction increase to 5.

The equalize-payments criteria selects buyer-optimal payoffs that minimize the maximal payment across all agents, and have the effect of equalizing agent payments. The method is illustrated below on Examples 1–4, with the simple algorithms described in the Appendix to determine the tight core- and recursive-core constraints.

example 1 Core constraints require $\{\pi_1 \leq 10, \pi_3 \leq 5$, and recursive-core constraints require, in addition, that $\pi_1 \leq 0$. Taking all recursive-core constraints, and with EQUAL-PAY, we have:

$$\begin{aligned} \min_{\pi} \ & \max\{10 - \pi_1, 25 - \pi_3\} \\ \text{s.t.} \quad & \pi_1 \leq 0, \\ & \pi_3 \leq 5 \end{aligned}$$

with $\pi_0 = 35 - \pi_1 - \pi_3$ and $\pi_2 = \pi_4 = \pi_5 = 0$. The solution, $\pi^* = (30, 0, 0, 5, 0, 0)$, corresponds with the outcome of the ascending-proxy auction.

example 3 Core constraints require $\{\pi_1 \leq 6, \pi_2 \leq 6, \pi_1 + \pi_2 \leq 6\}$, and recursive-core constraints require, in addition, that $\{\pi_1 \leq 0, \pi_2 \leq 0\}$. Ignoring the recursive-core constraints, and with EQUAL-PAY, we have:

$$\begin{aligned} \min_{\pi} \ & \max\{8 - \pi_1, 8 - \pi_2\} \\ \text{s.t.} \quad & \pi_1 \leq 6, \\ & \pi_2 \leq 6, \\ \pi_1 + {} & \pi_2 \leq 6 \end{aligned}$$

with $\pi_0 = 16 - \pi_1 - \pi_2$ and $\pi_3 = 0$. The solution, $\pi^* = (10, 3, 3, 0)$, corresponds with the outcome of the ascending-proxy auction.

example 4 Core constraints require $\{\pi_1 \leq 14, \pi_2 \leq 8, \pi_1 + \pi_2 \leq 14\}$, and recursive-core constraints require, in addition, that $\{\pi_1 \leq 6, \pi_2 \leq 0\}$. Ignoring the recursive-core constraints, and with EQUAL-PAY, we have:

$$\begin{aligned} \min_{\pi} \ & \max\{16 - \pi_1, 8 - \pi_2\} \\ \text{s.t.} \quad & \pi_1 \leq 14, \\ & \pi_2 \leq 8, \\ \pi_1 + {} & \pi_2 \leq 14 \end{aligned}$$

with $\pi_0 = 24 - \pi_1 - \pi_2$, and $\pi_3 = 0$. The solution, $\pi^* = (10, 11, 3, 0)$, corresponds with the outcome of the ascending-proxy auction.

Examples 3 and 4 provide good support for the match between the EQUAL-PAY selection criteria and the outcome selected by the ascending-proxy auction. Notice, though, that this analysis does not suggest a method to determine the appropriate recursive constraints to include. All are included in Example 1, but none are included in Examples 3 and 4.

However, Example 2 shows that there can sometimes be *no* subset of recursive-core constraints that characterizes the outcome of the auction, at least when in combination with the equalize-payments selection criteria.

example 2 Core constraints require $\{\pi_1 \leq 10, \pi_3 \leq 5\}$ and recursive-core constraints require, in addition, $\pi_1 \leq 0$. Consider the following two possibilities. First, take just the core constraints:

$$\begin{aligned} &\min_{\pi} \max\{10 - \pi_1, 25 - \pi_3\} \\ \text{s.t.} \quad &\pi_1 \leq 10, \\ &\pi_3 \leq 5 \end{aligned}$$

and set $\pi_0 = 35 - \pi_1 - \pi_3$, $\pi_2 = \pi_4 = \pi_5 = 0$. This formulation gives $\pi^* = (20, 10, 0, 5, 0, 0)$, which are the VCG payoffs. Second, also include the recursive-core constraint:

$$\begin{aligned} &\min_{\pi} \max\{10 - \pi_1, 25 - \pi_3\} \\ \text{s.t.} \quad &\pi_1 \leq 0, \\ &\pi_3 \leq 5 \end{aligned}$$

with $\pi_0 = 35 - \pi_1 - \pi_3$, $\pi_2 = \pi_4 = \pi_5 = 0$. This formulation gives $\pi^* = (30, 0, 0, 5, 0, 0)$. Neither payoff vector corresponds with that of the ascending-proxy auction, which implements $\pi = (29, 1, 0, 5, 0, 0)$.

Based on this example it would appear that the characterization can not be as simple as some subset of recursive-core constraints, coupled with a selection criteria to choose a buyer-optimal outcome. How can such an approach ever generate the payoff, $\pi_1 = 1$, that is generated by the ascending-proxy dynamics? Rather, the precise dynamics in the ascending-proxy auction influence the coalitions that must compete along the implementation path, and make this goal of finding a direct specification of the core outcome that is implemented within the auction difficult.

Rather than pursue this goal of direct characterization any further, in Section 5 we propose a "semi-direct" method to implement the outcome, given agent valuations, that avoids the incremental-bidding process of the indirect ascending-proxy implementation, but without providing any additional economic insight into the exact core point implemented by the auction.

4.2 Buyer-Optimal Core Outcome

Before leaving the discussion of indirect implementations of core outcomes, it is useful to present a simple variation on the ascending-proxy design. With this variation, which is suggested from analysis of *i*BEA [PU02], the auction will in fact always implement a buyer-optimal core outcome. This is useful because the auction retains the robustness of ascending-proxy against shills and joint-deviations, but also brings truthful bidding into equilibrium (and removes the bargaining problem) whenever agents-are-substitutes and without requiring the stronger condition of buyer-submodular values.

Notice that the useful robustness properties of the ascending-proxy equilibrium to joint deviations accrue from the *bidder monotonicity* property, and in turn follow from implementing (buyer-optimal?) payoffs in the core. The particular core outcome that is implemented in ascending-proxy, and intermediate between buyer-optimal core and buyer-optimal recursive-core points, is not necessary for these properties.

The ascending-proxy auction is unchanged, except for a slight variation in the method to compute final payoffs at termination. At termination, let π^* denote the final payoffs, $\mathcal{I}^*$ denote the set of winning agents, and $\pi_0^*(L)$, for $L \subseteq \mathcal{I} \setminus 0$, denote the *maximal payoff to the seller from bids from agents in set* L, or simply the value of the solution to the winner-determination based on the final bids of agents in L.

Definition 10 (core-adjust). *At termination, compute the adjustments, Δ^*, that solve the following linear program:*

$$\max_{\Delta} \sum_{i \in \mathcal{I} \setminus 0} \Delta_i \qquad \text{(CORE-ADJUST)}$$
$$\text{s.t.} \quad \sum_{i \in K} \Delta_i \leq \pi_0^* - \pi_0^*(\mathcal{I} \setminus \{K \cup 0\}), \quad \forall K \subseteq \mathcal{I}^*$$
$$\Delta_i \geq 0, \quad \forall i \in \mathcal{I}$$
$$\Delta_i = 0, \quad \forall i \notin \mathcal{I}^*$$

and implement final payoffs $\pi^* + \Delta^*$ to agents.

Notice that implementing this adjustment can be expensive, because in the worst-case the auction must solve an additional number of winner-determination problems that is *exponential* in the number of winning agents. In practice, experimental results suggest that these adjustments can be computed entirely over cached solutions along the winner-determination path during the auction [PU00b].

Proposition 2. *[PU02] The adjusted payoffs at the end of ascending-proxy implement a buyer-optimal core outcome with respect to reported agent valuations.*

Proof. All that is required for the proof is that agents follow MBR strategies, and that the payoffs at the end of ascending-proxy are somewhere in the core. The result is proved as Lemma 1 [PU02], in the context of third-order CE prices in *i*Bundle, which correspond exactly with the final MBR bids in ascending-proxy. □

Of course, when agents-are-substitutes, there is a unique buyer-optimal core point, which corresponds to the VCG payoffs. This slightly relaxes the conditions under which truthful reporting is an equilibrium of the auction, and under which there is no bargaining problem.

Theorem 1. *Suppose that the agents-are-substitutes condition holds on agent values. Then truthful reporting is a Nash equilibrium profile of the ascending-proxy auction and leads to the VCG payoffs.*

In fact, when agents-are-substitutes, the adjustments can be computed with the simpler adjustment method, $(VCG - ADJUST)$, described in the next section. In addition, with buyer-submodular values, the auction already terminates with VCG payoffs and the optimal adjustments are exactly zero because constraints $\pi_0^* = \pi_0^*(\mathcal{I} \setminus j)$ for all $j \in \mathcal{I}^*$.

An interesting open question that arises from the analysis in this section is the problem of finding a direct-characterization of the buyer-optimal point that is selected in this slightly modified ascending-proxy auction when agents-are-substitutes fails. It is even possible that a direct characterization of the outcome implemented at this adjusted payoff vector could lead to a direct characterization of the outcome of the current ascending-proxy auction!

4.3 VCG Outcomes

Finally, a quick comment about an indirect implementation of the VCG payoffs. Indirect mechanisms are often important, for example because of the high costs of preference-elicitation in direct mechanisms such as the VCG. Although the VCG outcome is susceptible to joint deviations and shill bidding [AM02], it does at least completely solve the bargaining problem which might in itself cause significant loss in efficiency and revenue, though miscoordination between agents.

In fact, one can view the VCG payoffs as the buyer-optimal payoffs subject to a relaxation of the core conditions to $\pi_0 = w(\mathcal{I}) - \sum_{i \in \mathcal{I} \setminus 0} \pi_i$ and

$$\pi_i \leq w(\mathcal{I}) - w(\mathcal{I} \setminus i), \quad \forall i \in \mathcal{I} \setminus 0$$

where all constraints that apply to the marginal product of coalitions of winning agents are dropped. With these unilateral constraints there is always a unique buyer-optimal payoff vector, at least in a one-sided combinatorial auction with no budget-balance problems.

All that one needs to implement the VCG payoffs in an indirect ascending-proxy like mechanism is to ensure that the auction terminates not just with payoffs that are in the core, but also with payoffs that are in the universal-core (4). As before, let π^* denote the final payoffs, $\mathcal{I}^*$ denote the set of winning agents, and $\pi_0^*(L)$ denote the value of the solution to the winner-determination based on the final bids of agents in $L \subseteq \mathcal{I} \setminus 0$.

Definition 11 (vcg-adjust). *At termination, compute the adjustments,*

$$\Delta_i^* = \pi_0^* - \pi_0^*(\mathcal{I} \setminus i), \quad \forall i \in \mathcal{I}^* \qquad \text{(VCG-ADJUST)}$$

with $\Delta_i^* = 0$ for all $i \neq \mathcal{I}^*$. Implement final payoffs of $\pi^* + \Delta^*$ to agents.

This adjustment is computationally easier to compute than (CORE-ADJUST), just requiring a solution to the winner-determination problem with each winning-agent removed from the auction.

Proposition 3. *[PU02, theorem 6] Suppose that the ascending-proxy auction terminates with payoffs in the universal-core with respect to reported valuations. Then, the adjusted payoffs at the end of ascending-proxy implement the VCG payoffs with respect to reported valuations.*

Proof. The universal-core property provides $\pi_0^* = w(\mathcal{I}) - \sum_{i \in \mathcal{I} \setminus 0} \pi_i^*$ and also $\pi_0^*(\mathcal{I} \setminus j) = w(\mathcal{I} \setminus j) - \sum_{i \in \mathcal{I} \setminus \{0,j\}} \pi_i^*$ for all $j \in \mathcal{I}^*$. Then, we have adjusted payoff, $\pi_{\text{adj},i} = \pi_i^* - (\pi_0^* - \pi_0^*(\mathcal{I} \setminus i)) = w(\mathcal{I}) - w(\mathcal{I} \setminus i) = \pi_{\text{vcg},i}$, for all $i \in \mathcal{I}^*$. □

An indirect implementation of the VCG outcome follows from an extension to *i*Bundle [PU02], that drives an ascending-proxy into an outcome in the universal-core. The auction is unchanged until the round in which it would terminate as currently described. The auction then continues to run ascending-proxy methods from the termination point, removing each winning agent in turn, until the auction reaches a core solution for both the complete set of agents and with each winning agent removed. This is a universal-core outcome, at which point the adjustments are computed and the VCG payoffs implemented. Call this extended auction, with final payoff adjustments computed as (VCG-ADJUST), the *extended ascending-proxy* auction.

Theorem 2. *Truthful reporting is a Nash equilibrium profile of the extended ascending-proxy auction and leads to the VCG outcome.*

Variations of *i*Bundle Extend & Adjust (*i*BEA) are also proposed that maintain ask prices and introduce non-anonymous prices dynamically [PU02].

In fact, it is *necessary* that payoffs are in the universal-core for (VCG-ADJUST) to implement VCG payoffs [PU02, theorem 2]. In addition, the adjusted payoffs computed with (VCG-ADJUST) at the end of ascending-proxy are always (a) weakly dominated by the VCG payoffs and in turn always (b) weakly dominate the payoffs computed with (CORE-ADJUST). From this, we learn a little more about the characterization of the outcome of the ascending-proxy auction. When agents-are-substitutes the adjusted payoffs computed in (VCG-ADJUST) must equal the VCG payoffs, and payoffs are necessarily in the universal-core.

Proposition 4. *When reported valuations in the ascending-proxy auction satisfy agents-are-substitutes the auction terminates with payoffs in the universal-core with respect to the reported valuations.*

This complements the earlier characterization that the ascending-proxy auction terminates with buyer-optimal recursive-core payoffs when values are buyer-submodular.

5 Direct Implementations

In this section we propose a variety of direct implementations of buyer-optimal core outcomes, that retain the useful robustness of the ascending-proxy auction to shill bidding and joint deviations. While deemphasizing the goal of characterizing the core outcome implemented in the current ascending-proxy design, the goal is to open up a wider discussion on the appropriate selection criteria that an auction should employ to address the bargaining problem. An interesting open question that follows form the direct formulations is the reverse of that addressed in the previous section: *are there indirect mechanisms, such as ascending-price auctions, that implement the outcomes?*

One can imagine that these methods attempt to solve the bargaining problem for agents in settings in which they fail to completely solve the problem themselves. The methods act as a coordination device, something at which mechanisms have a natural advantage over agents because the mechanism gets to play *ex post*, after every agent has committed to a particular strategy.

5.1 Core Outcomes

This section presents a variety of formulations to select buyer-optimal core outcomes. These all implement the VCG outcome when it is in the core, but provide the additional bidder monotonicity properties that have been shown to be useful when the VCG outcome is outside the core. The selection criteria, which chooses a particular buyer-optimal core payoff vector, distinguishes the proposals.

Let (CORE) represent the following constraints:

$$\begin{aligned}
\pi_0 &= w(\mathcal{I}) - \sum_{i \in \mathcal{I} \setminus 0} \pi_i \\
\sum_{i \in K} \pi_i &\leq w(\mathcal{I}) - w(\mathcal{I} \setminus K), \quad \forall K \subset \mathcal{I}^*, 0 \notin K \\
\pi_i &\geq 0, \quad \forall i \\
\pi_i &= 0, \quad \forall i \notin \{\mathcal{I}^* \cup 0\}
\end{aligned}$$

where $\mathcal{I}^*$ is the set of winning agents, that receive a non-empty bundle in the efficient allocation.

First, the *equalize-payments* criteria, that provided a reasonable match to the ascending-proxy outcomes, is one possibility.

$$\begin{aligned}
&\min_{\pi} \max_{i \in \mathcal{I} \setminus 0} [v_i(S_i^*) - \pi_i] \qquad \text{(EQUAL-PAY)} \\
&\text{s.t.} \quad \text{(CORE)}
\end{aligned}$$

where S^* denotes the efficient allocation. The (EQUAL-PAY) criteria attempts to equalize the payments made by each agent.

Second, the *threshold* criteria from Parkes et al. [PKE01b, PKE01a], which minimizes the *ex post* maximal difference from VCG payoffs across agents.

$$\min_{\pi} \max_{i \in \mathcal{I} \setminus 0} [\pi_{\text{vcg},i} - \pi_i] \qquad \text{(THRESHOLD)}$$
$$\text{s.t.} \quad \text{(CORE)}$$

This selection criteria was previously introduced in the context of selecting payoffs that minimize the maximal difference from VCG payoffs in a combinatorial exchange setting subject to budget-balance constraints. Budget-balance is not binding with VCG-based selection criteria in the single-sided combinatorial auction setting, but core constraints are.

In the combinatorial exchange setting the optimal solution to this threshold formulation is characterized with a simple rule, $\pi_i = \max(0, \pi_{\text{vcg},i} - C)$, with a value $C \geq 0$ selected to give budget-balance. Hence the name "threshold", because the rule assigns payoff to agents that have VCG payoffs above the threshold value, C. An interesting open question is to investigate whether there are similar closed-form solutions to particular formulations of the core payoff-division problem.

Parkes *et al.* also considered additional VCG-payoff based rules, that correspond with a simple selection-criteria for budget-balanced constraints, but have no obvious direct correspondence with richer core constraints. Examples of rules included:

large taking the agents in order of decreasing VCG payoff, and assigning as much payoff to the agent as possible (up to its VCG payoff), subject to constraints. This can be viewed as a heuristic method to solve the core payoff-division problem with selection criteria $\max_\pi \sum_i \pi_{\text{vcg},i}\pi_i$.

small taking the agents in order of increasing VCG payoff, and assigning as much payoff to the agent as possible (up to its VCG payoff), subject to constraints. This can be viewed as a heuristic method to solve the core payoff-division problem with selection criteria $\max_\pi \sum_i \pi_i/\pi_{\text{vcg},i}$.

fractional provide each agent with the same fraction of its VCG payoff, subject to constraints. This can be viewed as a heuristic method to solve the core payoff-division problem with selection criteria $\max_\pi \min_i \pi_i/\pi_{\text{vcg},i}$.

Based on analytic and experimental results in the combinatorial exchange setting, the Threshold rule appeared the most promising in terms of its equilibrium efficiency properties. The experiments computed a restricted Bayesian-Nash equilibrium under each rule [PKE01a]. Intuitively, the Threshold rule solves the bargaining problem by *providing payoff to agents that could have gained a lot from taking an alternative bargaining position.*

It is worthwhile to briefly consider the combinatorial exchange setting, in which the core is often empty. That the core is empty follows from the competitive equilibrium analysis of the two-sided package assignment model [BO02]. This means that in the context of a combinatorial exchange one cannot simply impose core constraints. Instead, one can impose relaxed core constraints, such as the unilateral VCG-type core constraints, $\pi_i \leq w(\mathcal{I}) - w(\mathcal{I} \setminus i)$. This is the approach taken in Parkes *et al.* As discussed in July's FCC spectrum-design meeting, one could also impose core constraints for a subset of participants, such as for agent representing within an exchange and then use a selection criteria such as Threshold to clear the exchange.

5.2 Ascending-Proxy Outcome

The outcome of the ascending-proxy auction (with respect to reported valuations) is always in the core, and in the universal-core if agents-are-substitutes holds, and in the recursive-core if buyer-submodular values holds. We also know that the payoffs always buyer weakly-dominate all buyer-optimal recursive-core payoffs, from which it follows that the auction implements the VCG payoffs when buyer-submodular values holds. The selection criteria used within the auction also seems somewhat consistent with the equalize-payments criteria. However, the attempts in Section 5 failed to provide a complete and direct characterization of the outcome of the auction.

Another motivation for a direct characterization of the outcome of the ascending-proxy auction, other than to gain economic insight into the outcome selected by the auction, was to provide an efficient method to implement the outcome as a direct-revelation mechanism. To address these computational questions we fall back in this section on a proposal for a *semi-direct* implementation of the outcome of the ascending-proxy auction, which should at least adequately handle problems in which the agent-proxy language has a rich preference language (e.g. Milgrom's "scenario-based" language) without exploding the XOR representation at the proxy-auction interface.[9]

5.3 Semi-Direct Implementation

We propose a two-step accelerated implementation of the ascending-proxy auction. The first step is to compute the *interesting coalitions*, and the *interesting bundles* for each agent. The interesting coalitions are those subcoalitions of agents that might be involved in dynamic price competition during the auction. Let C^* denote the set of interesting coalitions. We know that anytime coalition $x \in C^*$ is winning in the auction, that the coalition is winning with the same allocation. Let T_i denote the set of interesting bundles for agent i, and compute the bundle that corresponds with coalition $x \in C^*$, with $i \in x$, as the bundle agent i receives in the solution to $w(x)$. C^* always includes all the singleton agents, call these the *trivial coalitions.* The interesting bundle that correspond with the singleton coalition $\{i\}$, is simply $\max_S v_i(S)$. Call all coalitions with more than one agent a *non-trivial* coalition.

The Appendix presents a simple algorithm to determine the interesting coalitions and bundles. It is useful that this first step of the semi-direct implementation determines "reduced valuations" for agents, which provide sufficient information to compute the outcome of the auction. The reduced valuation function is simply an XOR valuation defined over an agent's values for bundles in its interesting set. The interesting and nontrivial coalitions for Example 1, as derived in the Appendix, are coalitions {13,12,45}, and the interesting bundles for this example are simply $(\{AB\}, \{CD\}, \{CD\}, \{AC\}, \{BD\})$, for agents $1, \ldots, 5$ respectively.

We propose two alternatives for the second step.

run ascending-proxy on the reduced valuations. one can simply run the ascending-proxy auction on the reduced valuations. this has the advantage of providing transparency, while preventing the explosion in the size of the XOR bid lists that can occur in the basic indirect implementation.

run a staged variation on the ascending-proxy with the reduced valuations. alternatively, one can run an accelerated, or staged, proxy auction based on the reduced valuations. the acceleration provided in this approach will depend on the size of agent interesting sets (smaller is better) and on the spread in the valuations in an agent's interesting set (smaller is better).

The staged ascending-proxy auction runs in stages, $t \geq 1$, that correspond to a contiguous sequence of regular rounds. At the start of each stage, agent i has a current payoff, π_i^t, which is

[9] I have not yet proved this algorithm terminates with the outcome of the ascending-proxy auction. A useful next step is to compare my proposal with that in the Hoffman *et al.* proposal [DHM$^+$].

initially $\pi_0^1 = 0$ for the seller, and $\pi_i^1 = \max_{S \in T_i} v_i(S)$ for all $i \in \mathcal{I} \setminus 0$. There is a set of *active agents* in each round, $A^t \subseteq \mathcal{I} \setminus 0$, which denotes the agents with $\pi_i^t > 0$. Let $\text{MBR}_i(\pi_i^t)$ denote the set of interesting bundles in agent i's best-response set at the start of stage t. This is computed as

$$\text{MBR}_i(\pi_i^t) = \{S \mid S \in T_i,\ v_i(S) - \pi_i^t \geq 0\}$$

In addition, let δ_i^t denote the MBR *slack* for agent i at the start of stage t. This is computed as

$$\delta_i^t = \min\left[\pi_i^t, \{\pi_i^t - v_i(S) \mid S \notin \text{MBR}_i(\pi_i^t), S \in T_i\}\right]$$

and represents the maximal possible decrease in π_i^t that will leave the agent's MBR set unchanged. This information is vital for providing an accelerated implementation of the auction. An agent is *active* in round t while $\delta_i^t > 0$.

Given MBR information, each stage of the accelerated auction is implemented as an LP. Let k index into the interesting coalitions, C^*. Let C^t denote the *active* coalitions at the start of stage t. These are the coalitions that include at least one active agent, *and* for which the associated interesting bundles are receiving bids in agent MBR sets. A coalition is not active until every agent is submitting the relevant bid. Moreover, all *dominated* coalitions are pruned from C^t, where x is dominated by any $x' \supset x$.

Given the active coalitions, the LP is formulated over decision variables, $x = \{x_k \ : \ k \in C^t\}$, where $x_k \geq 0$ is the *bidding-share* for active coalition k, and interpreted as the minimal drop in payoff to agents bidding in coalition k during the stage. At the end of the stage, agent payoffs are decreased by the maximal bidding-share, x_k, over all coalitions in which they were bidding during the round. Formally:[10]

$$\pi_0^{t+1} = \max_x \min_{k \in C^t} \{V_k\} \tag{STAGE}$$

$$\text{s.t.} \quad V_k \geq \pi_0^t + \sum_{i \in C^*(k),\ bid^t(k,i)} \left[\max_{k' \ : \ i \in C^*(k'), k' \in C^t} x_{k'} \right], \quad \forall k \in C^t \tag{2}$$

$$\delta_i^t \geq \max_{k' \ : \ i \in C^*(k'), k' \in C^t} x_{k'}, \qquad \forall i \in \mathcal{I} \setminus 0, \text{with } \delta_i^t > 0 \tag{3}$$

$$x_k \geq 0, \qquad \forall k \in C^t$$

where agent i bids, $bid^t(k, i)$, in coalition k during stage t if

$$bid^t(k,i) = \left(\exists k' \in C^t \ \cdot \ i \notin C^*(k')\right) \text{ and } \delta_i^t > 0$$

in words, there must be another active coalition that does not include i, and the agent must also still be active.

The objective (STAGE) is to find adjustments to the payoffs of agents that maximize the minimal revenue to the auctioneer from all active and interesting coalitions. Constraints (2) ensure that V_k for active coalition k evaluates to the current revenue from the coalition, along with whatever increases in that revenue accrue due to lower payoffs to agents within the coalition. Payoff changes are accounted for all agents within the coalition that are in competition with at least one other active coalition, such that $bid^t(k, i)$ holds. Moreover, the payoff change associated with agent i is computed as the maximal over the bidding shares, x_k, in all active coalitions that include the agent and in which the agent is bidding. Constraints (3) ensure that the payoff change for agent i is within its MBR slack.

[10] A complete LP formulation would need to formulate away the remaining max terms, etc.

In fact, the LP is solved a number of times in each round, as many times as is required to implement a lexicographically minimal solution over $\{V_k\}$ for $k \in C^t$. First, solve (STAGE), to compute $\{\hat{V}_k\}$, with $V^t = \min_{k \in C^t} \hat{V}_k$. Formulate and solve a series of LPs, indexed $m = 1, \ldots$, and label them (STAGE$_m$). Associate $m = 1$ with the initial problem, already solved as (STAGE). Across stages we build-up a set of coalitional values that are fixed, with $F^m \subseteq C^t$. Initially, set $F^1 = \emptyset$. Let V^m denote the value of (STAGE$_m$), with the value of each individual coalition, k, denoted V_k^m. At the end of (STAGE$_m$), construct $F^{m+1} = F^m \cup \{k \ : \ k \in C^t \setminus F^m, V_k^m = V^m\}$. To solve (STAGE$_m$), for $m > 1$, formulate the objective as $\max_x \min_{k \in C^t \setminus F^m} \{V_k\}$, with constraints (2) for $k \in F^m$ replaced with $V_k = V_k^{m-1}$. Continue until $F^m = C^t$.

At the end of the stage the agent payoffs are adjusted, with

$$\pi_i^{t+1} = \pi_i^t + \max_{k' \ : \ i \in C^*(k'), k' \in C^t} x_{k'}$$

and the auctioneer's revenue is updated to $\pi_0^{t_1}$ as computed by (STAGE). The next round is initialized by updating the MBR bid sets and the MBR slack values, and updating the active coalitions. The auction terminates whenever all active agents are included together in an active coalitions. The outcome associated with this winning coalition is implemented, along with payments that correspond to the final agent payoffs.

Example 1. The non-trivial interesting coalitions are $C^* = \{12, 13, 45\}$, and the interesting bundles are $T = (\{AB\}, \{CD\}, \{CD\}, \{AC\}, \{BD\})$ for agents $1, \ldots, 5$. The initial payoffs are (0,10,20,25,10,10), and the initial MBR sets are $\text{MBR}_i(\pi_i^1) = T_i$, with δ_i^1 equal to agent valuations. All agents are active, and the set of active coalitions, $C^1 = \{12, 13, 45\}$, with coalitions {1,2,3,4,5} all pruned. Index the coalitions in C^1 with $k \in \{1, 2, 3\}$, such that $C^*(1) = 12$, $C^*(2) = 13$ and $C^*(3) = 45$. The stage LP is formulated as:

$$\begin{aligned}
\pi_0^2 = \max_{x_1, x_2, x_3} & \min \{V_1, V_2, V_3\} \\
\text{s.t.} \quad & V_1 \geq 0 + \max(x_1, x_2) + x_1 && (12) \\
& V_2 \geq 0 + \max(x_1, x_2) + x_2 && (13) \\
& V_3 \geq 0 + x_3 + x_3 && (45) \\
\max(x_1, x_2) & \leq 10 && (a1) \\
& x_1 \leq 20 && (a2) \\
& x_2 \leq 25 && (a3) \\
& x_3 \leq 10 && (a4) \\
& x_3 \leq 10 && (a5) \\
x_1, x_2, x_3 & \geq 0
\end{aligned}$$

with $\{x_1, x_2, x_3\}$ to denote the bidding-shares computed for each coalition. Labels (12,13) and (45) denote coalition values, for 12, 13 and 45, and labels (a1,…,a5) denote agents, for agents 1 through 5. Solving, we get $x^* = (10, 10, 10)$ and $\pi_0^2 = 20$. All values, V_k, are tight and there is no lexicographical ordering required. The agent payoffs adjust to $\pi^2 = (20, 0, 10, 15, 0, 0)$.

Continuing, in the next stage the MBR slack values adjust to $\delta^2 = (0, 10, 15, 0, 0)$, and only agents 2 and 3 are still active. The only active coalitions are $C^2 = \{12, 13\}$. The stage LP is formulated as:

$$\pi_0^3 = \max_{x_1,x_2} \min\{V_1, V_2\}$$

$$\text{s.t.} \quad V_1 \geq 20 + x_1 \qquad (12)$$

$$V_2 \geq 20 + x_2 \qquad (13)$$

$$x_1 \leq 10 \qquad (a2)$$

$$x_2 \leq 15 \qquad (a3)$$

$$x_1, x_2 \geq 0$$

Agent 1 is not bidding in either coalition 12 or coalition 13, because there are no active coalitions that do not include agent 1. Rather, agent 2 is in competition with agent 3. Notice that the bidding-share, x_1, in coalition 12 is not allocated to agent 1, but only agent 2, and similarly for share x_2 in coalition 13, which is only allocated to agent 3. An optimal solution sets $x_1^* = 10$, with $10 \leq x_2 \leq 15$. Lexicographical ordering, formulating a second optimization problem with $V_1 = 10$ and minimizing V_2 gives $x_2^* = 10$. The new payoffs are $\pi^3 = (30, 0, 0, 5, 0, 0)$, and moreover the only active coalition is $C^3 = \{13\}$, and the auction terminates. This is the outcome of the ascending-proxy auction.

Example 2. The set-up is the same as for Example 1, except that the initial payoffs are (0,10,20,25,8,10). The LP for stage 1 is formulated as:

$$\pi_0^2 = \max_{x_1,x_2,x_3} \min\{V_1, V_2, V_3\}$$

$$\text{s.t.} \quad V_1 \geq 0 + \max(x_1, x_2) + x_1 \qquad (12)$$

$$V_2 \geq 0 + \max(x_1, x_2) + x_2 \qquad (13)$$

$$V_3 \geq 0 + x_3 + x_3 \qquad (45)$$

$$\max(x_1, x_2) \leq 10 \qquad (a1)$$

$$x_1 \leq 20 \qquad (a2)$$

$$x_2 \leq 25 \qquad (a3)$$

$$x_3 \leq 8 \qquad (a4)$$

$$x_3 \leq 10 \qquad (a5)$$

$$x_1, x_2, x_3 \geq 0$$

with $\{x_1, x_2, x_3\}$ to denote the bidding-shares computed for each coalition. The one change is in the RHS of (a4), which is adjusted to the smaller value of agent 4. An optimal initial solution sets $x^* = (7, 9, 8)$. However, we can then fix $V_3 = 16$ and $V_1 = 16$, and reoptimize to minimize the value of V_2. This gives solution $x^* = (8, 8, 8)$. The adjusted agent payoffs are $(16, 2, 12, 17, 0, 2)$.

Continuing, in the next stage the MBR slack values adjust to $\delta^2 = (2, 12, 17, 0, 2)$, and all agents except 4 are still active. The active coalitions remain $C^2 = \{12, 13, 45\}$, with 45 retained because

5 is still active. The stage LP is formulated as:

$$\begin{aligned}
\pi_0^3 = \max_{x_1,x_2,x_3} & \min\{V_1, V_2, V_3\} \\
\text{s.t.} \quad & V_1 \geq 16 + \max(x_1, x_2) + x_1 && (12)\\
& V_2 \geq 16 + \max(x_1, x_2) + x_2 && (13)\\
& V_3 \geq 16 + x_3 && (45)\\
\max(x_1, x_2) & \leq 2 && (a1)\\
& x_1 \leq 12 && (a2)\\
& x_2 \leq 17 && (a3)\\
& x_3 \leq 2 && (a5)\\
x_1, x_2, x_3 & \geq 0
\end{aligned}$$

Notice that no bidding-share is allocated to agent 4 in coalition 45 because agent 4 is no longer active. Similarly, no constraint is required to restrict x_3 to the MBR slack of agent 4. The optimal solution, again after lexicographical ordering, is $x^* = (1, 1, 2)$, from which the adjusted payoffs are $(18, 1, 11, 16, 0, 0)$.

At this stage, neither agents 4 or 5 are active, and $C^3 = \{12, 13\}$. As in Example 1, when we construct the LP for this stage, agent 1 does not receive any bidding-share because the agent is not in competition with any active coalition, and is not included in the MBR slack constraints. The LP for the final stage is:

$$\begin{aligned}
\pi_0^4 = \max_{x_1,x_2} & \min\{V_1, V_2\} \\
\text{s.t.} \quad & V_1 \geq 18 + x_1 && (12)\\
& V_2 \geq 18 + x_2 && (13)\\
& x_1 \leq 11 && (a2)\\
& x_2 \leq 16 && (a3)\\
x_1, x_2 & \geq 0
\end{aligned}$$

After lexicographical ordering the optimal solution is $x^* = (11, 11)$, and the adjusted payoffs are $\pi^3 = (29, 1, 0, 5, 0, 0)$. The algorithm terminates because the only active coalition is 13, and we implement the outcome of the ascending-proxy auction.

Example 3. The non-trivial interesting coalitions are $C^* = \{12\}$, and the interesting bundles are $T = (\{A\}, \{B\}, \{AB\})$ for agents 1,2 and 3. Notice that in determining interesting bundles we are free to break ties within any particular interesting coalition (e.g. 12) at random. Here we assign A to agent 1 and B to agent 2. Initial payoffs are (0,8,8,10), and the initial MBR sets are $\text{MBR}_1 = \{A\}, \text{MBR}_2 = \{B\}$ and $\text{MBR}_3 = \{AB\}$. The slack values are $\delta^1 = (8, 8, 10)$.

All agents are initially active, and the set of active coalitions is $C^1 = \{12, 3\}$, with coalitions 1 and 2 pruned by 12. Index the coalitions in C^1 with $k \in \{1, 2\}$, such that $C^*(1) = 12$, $C^*(2) = 3$.

The stage LP is formulated as:

$$\pi_0^2 = \max_{x_1,x_2} \min\{V_1, V_2\}$$

$$\text{s.t.} \quad V_1 \geq 0 + x_1 + x_1 \tag{12}$$

$$V_2 \geq 0 + x_2 \tag{3}$$

$$x_1 \leq 8 \tag{a1}$$

$$x_1 \leq 8 \tag{a2}$$

$$x_2 \leq 10 \tag{a3}$$

$$x_1, x_2 \geq 0$$

with $\{x_1, x_2\}$ to denote the bidding-shares computed for each coalition. Solving, and lexicographically minimizing, we get $x^* = (5, 10)$, and adjusted payoffs of $\pi^2 = (10, 3, 3, 0)$. The only active coalition that remains is 12, and the auction terminates with the outcome of the ascending-proxy auction.

Example 4. The initial state is the same as in Example 3, except that initial payoffs are (0,16,8,10), and the initial slack values are $\delta^1 = (16, 8, 10)$. The set of active coalitions is $C^1 = \{12, 3\}$, with coalitions 1 and 2 pruned by 12. The stage LP is formulated as:

$$\pi_0^2 = \max_{x_1,x_2} \min\{V_1, V_2\}$$

$$\text{s.t.} \quad V_1 \geq 0 + x_1 + x_1 \tag{12}$$

$$V_2 \geq 0 + x_2 \tag{3}$$

$$x_1 \leq 16 \tag{a1}$$

$$x_1 \leq 8 \tag{a2}$$

$$x_2 \leq 10 \tag{a3}$$

$$x_1, x_2 \geq 0$$

with $\{x_1, x_2\}$ to denote the bidding-shares computed for each coalition. Solving, and lexicographically minimizing, we get $x^* = (5, 10)$, and adjusted payoffs of $\pi^2 = (10, 11, 3, 0)$. The only active coalition that remains is 12, and the auction terminates with the outcome of the ascending-proxy auction.

6 Open Questions

- Is there a direct characterization of either the ascending-proxy auction outcome, or the variation on the ascending-proxy auction that adjusts payoffs and implements a buyer-optimal core outcome?
- If the Threshold selection criteria proves interesting, can we provide an indirect mechanism that implements the same outcome? What about other selection criteria, VCG-based or otherwise. For example, is there an indirect implementation of the equalize-payments core payoff division rule?
- Check that the proposed semi-direct algorithm to implement the outcome of the ascending-proxy auction terminates with the VCG payoffs in the special-case of buyer-submodular values, and with a universal-core outcome in the special-case of agents-are-substitutes values. More generally, prove the staged implementation method is correct.

- What is the right rule to solve the bargaining problem that the mechanism inherits from agents when they fail to solve the problem amongst themselves?
- Is the Bayesian-Nash equilibrium of the Threshold core payoff-division method better (in efficiency terms) than the other rules? Threshold reduces to the optimal double auction rule for Chatterjee & Samuelson's [CS83] seminal k-double auction model with one item and two agents, in which $k = 0.5$ maximizes the expected gains from trade for i.i.d. uniform values and agents play a linear equilibrium [MS83].

7 Conclusions

We have provided a partial characterization of the bargaining outcome implemented in the ascending-proxy auction, and proposed a number of criteria that one could use to select buyer-optimal core payoffs in a direct mechanism. We also proposed a staged implementation of the ascending-proxy auction, that might be particularly useful from a computational perspective in settings with rich bidder-proxy interfaces.

Appendix

Method to Compute Interesting Core Constraints

Consider a problem in which $\mathcal{I} = \{0, 1, \ldots, 5\}$, with winning agents, $\mathcal{I}^* = \{0, 1, 2\}$. In full, (ALT-CORE) requires a constraint for every subset $K \subset \mathcal{I}^* \setminus 0$. However, many of these can be pruned as the constraints are constructed. As an example, consider the following are two core constraints from (ALT-CORE):

$$\pi_1 \leq w(\mathcal{I}) - w(\mathcal{I} \setminus 1) \tag{4}$$

$$\pi_1 + \pi_2 \leq w(\mathcal{I}) - w(\mathcal{I} \setminus \{1, 2\}) \tag{5}$$

Notice that constraint (4) is dominated by (5), unless $w(\mathcal{I} \setminus 1) > w(\mathcal{I} \setminus \{1, 2\})$. This suggests an improved algorithmic method to construct the important core constraints. Let C enumerate all possible subsets of $\mathcal{I}^* \setminus 0$. Consider constraints for sets if C in increasing order. For any element, $K \in C$, solve $w(\mathcal{I} \setminus K)$. Let D denote the set of agents in this solution. Let $D' = \mathcal{I}^* \setminus \{K \cup D\}$. If $D \neq \emptyset$ then some of the agents that were winning are no longer winning, and $w(\mathcal{I} \setminus K) = w(\mathcal{I} \setminus \{K \cup D'\}$. Use this to construct the core constraint for $\{K \cup D'\}$, and prune away from set C all constraints corresponding with $K' \subset \{K \cup D'\}$ and $K \subseteq K'$.

Example. Consider the following trace, in which * indicates a winning agent. With all agents, 1, 2 and 3 win, giving 1*2*3*45. Start with 1, solve 234*5*. Prune $\{1, 12, 13\}$ from the core set, and include the constraint for 123. Go to 2, solve 1*34*5. Prune $\{2\}$ from the core set, and include the constraint for 23. Go to 3, solve 1*24*5, prune $\{3\}$ (23 already included). Done. Interesting core constraints included for sets $\{123, 23\}$.

Method to Compute Interesting Recursive Core Constraints

In putting together this note it was useful develop a fast method to compute interesting recursive-core constraints. The method seeks to avoid computing the core constraints for *all* subcoalitions.

Algorithm: maintain a stack V of coalitions, initially $V = [\mathcal{I}]$. also maintain a set of interesting coalitions, $C = \{\cdot\}$, initially empty. let $\mathcal{I}^*$ denote the winning agents. while $V \neq \emptyset$ pop a set x

from the stack. if $x \cap \mathcal{I}^* = \emptyset$, discard. otherwise, push x into the set of interesting coalitions, and solve for $w(x)$. let D denote the set of winners in this solution. if $D \cap \mathcal{I}^* \neq \emptyset$ then construct subsets $E = \{x \setminus j \ : \ \forall j \in D\}$ and push them onto stack V. Discard x. terminate the while loop when V is empty. Finally, use the algorithm to construct interesting core constraints can be used for each element in the list of interesting coalitions, reusing the computation already performed in this algorithm to avoid resolving the efficient allocation problems in each coalition. In running the algorithm one needs to be a little bit careful with ties, for example only discarding sets x with *no* solution that contain winning agents, and considering multiple solutions when branching.

Example. As an example, the trace of the algorithm for Example 1 looks something like the following. Use * to illustrate winning agents. Writing down the trace of popped coalitions, we first pop 1*23*45, which is interesting and then push 2345 and 1245 back onto the stack. Pop 1*2*45, interesting, and then push 245 and 145 onto the stack. Pop 14*5*, interesting, but then discard because neither {1,3} are now winning. Pop 2*45, but discard, not interesting. Pop 23*45, interesting, and push 245. Finally, discard 245. Done. At the end, the interesting sets are $\{12345, 1245, 145, 2345\}$, which lead eventually to constraints $\pi_1 \leq 0$ and $\pi_3 \leq 5$.

Method to Select Interesting Coalitions for Ascending-Proxy Price Dynamics

This subsection presents the method used to determine *reduced preferences* in the semi-direct, staged implementation of the ascending-proxy auction. We present a method to determine:

1. the set, C^*, of *interesting coalitions*, which identifies sets of agents that are involved in driving price dynamics during the auction

2. the set, T_i, of *interesting bundles*, for all agent $i \neq 0$, which are the bundles that agent i demands that fit in useful ways with demands from other agents.

This reduced information, which can be computed as a sequence of optimization problems based on agent valuations, provides sufficient information to implement the outcome of the auction. The interesting bundles are computed from the interesting coalitions. The following algorithm computes the interesting coalitions. The set of interesting coalitions will always include all singleton coalitions, we refer to these as *trivial* coalitions.

Algorithm: maintain a stack V of coalitions, initialized with $V = [\mathcal{I}]$, and initialize the set of interesting coalitions, $C^* = \{\{i\} \ : \ i \in \mathcal{I} \setminus 0\}$. while stack, $V \neq \emptyset$, pop an element x. if all subsets of x, including x, are in C^*, then drop x. Otherwise, solve $w(x)$, and let $D(x)$ denote the winning agents. Add $D(x)$ to the interesting set, unless it is already present. Whether or not x is interesting, branch on all winning agents in x, generating additional subsets $E = \{x \setminus j \ : \ \forall j \in D(x)\}$, and push all $x' \in E$ onto the stack as long as $|x'| > 1$, or unless there is already an identical set on the stack. Continue until the stack is empty.

Example. Consider Example 1. Here is a trace of the elements popped from the stack, with winning agents marked *. Initialize $C^* = \{1, 2, 3, 4, 5\}$, and $V = [12345]$. Pop 1*23*45, and add $\{13\}$ to C^*. Push 2345 and 1245 onto the stack. Pop 23*45, but do not add $\{3\}$ to C^* because it is already there. Push 245 onto the stack. Pop 1*2*45 and add $\{12\}$ to C^*. Push 145 onto the stack, but not 245 because it is already there. Pop 24*5* and add $\{45\}$ to C^*. Push 24 and 25 onto the stack. Also consider the tie, with 2*45, but 2 is already in C^*. Pop 14*5* put don't add $\{45\}$ to C because it is already there. Push 14 and 15 onto the stack. Pop 2*4, and but drop 24 without adding any new sets to C^*. Similarly for 25, 45, 14, and 15 which are all popped and dropped until the stack is empty. Done. At the end the non-trivial interesting coalitions are $C^* = \{13, 12, 45\}$.

Once we have the interesting coalitions it is easy to compute the interesting bundles for each agent. For each agent i, initialize $T_i = \emptyset$. Then, for each $x \in C^*$ with $i \in x$, add the bundle that is allocated to i in the solution to $w(x)$ to T_i. This generates the set of interesting bundles for that agent.

Example. Back to Example 1. We have $C^* = \{1, 2, 3, 4, 5, 13, 12, 45\}$. Now, take agent 1. Agent 1 is in interesting coalitions $\{1, 13, 12\}$, all times with bundle AB. So $T_1 = \{AB\}$. Agent 2 is in coalitions $\{2, 12\}$, with bundle CD, and $T_2 = \{CD\}$. Similarly, $T_3 = \{CD\}$, $T_4 = \{AC\}$ and $T_5 = \{BD\}$.

Proof of Core and CE Equivalence.

Lemma 1. *Payoffs are in the core in every competitive equilibrium and every set of core payoffs is supported in some competitive equilibrium.*

Proof. (CE $\Rightarrow$ *Core*) Consider allocation S, in competitive equilibrium with prices, p. Construct core payoffs as:

$$\pi_i = \max_{S' \subseteq \mathcal{G}} (v_i(S') - p_i(S')), \quad \forall i \in \mathcal{I} \setminus 0$$

with $\pi_0 = \sum_{i \in \mathcal{I} \setminus 0} p_i(S_i)$. To verify that these payoffs are in the core, first note that we have $\sum_{i \in \mathcal{I}} \pi_i = w(\mathcal{I})$, or feasibility, by LP duality. Now, to show $\pi_0 \geq w(L) - \sum_{i \in L \setminus 0} \pi_i$ for all $L \subseteq \mathcal{I}$, consider allocation, S_L, that maximizes the payoff (in terms of revenue) to the seller at prices p_i across all allocations to agents L. Let $\pi_0(S_L) = \sum_{i \in L} p_i(S_{L,i})$. By (CS2) we know that $\pi_0 \geq \pi_0(S_L)$, so it remains to show that $\pi_0(S_L) \geq w(L) - \sum_{i \in L \setminus 0} \pi_i$. But, prices p_i restricted to $i \in L$ define a feasible dual solution to CAP(L), with dual value $\pi_0(S_L) + \sum_{i \in L \setminus 0} \pi_i$, which is no less than $w(L)$ by weak duality.

(*Core* $\Rightarrow$ CE) Consider $\pi \in \mathit{Core}(\mathcal{I}, w)$. Construct allocation, $S^* = \max_{S \in \mathcal{X}} \sum_{i \in \mathcal{I} \setminus 0} v_i(S_i)$ and prices $p_i(S) = \max(0, v_i(S) - \pi_i)$ for all $S \subseteq \mathcal{G}$ and all $i \in \mathcal{I} \setminus 0$. Condition (CS1) holds for agents $i \neq 0$ by construction, since $\pi_i \leq v_i(S_i^*)$ in the core. Now, the seller's revenue from allocation S^* is simply its core payoff, because $\pi_0 = w(\mathcal{I}) - \sum_{i \in \mathcal{I} \setminus 0} \pi_i = \sum_{i \in \mathcal{I} \setminus 0} v_i(S_i) - \sum_{i \in \mathcal{I} \setminus 0} \pi_i = \sum_{i \in \mathcal{I} \setminus 0} p_i(S_i)$. Condition (CS2) follows. Suppose otherwise, that there is some allocation S' for which $\sum_{i \in \mathcal{I} \setminus 0} p_i(S'_i) > \pi_0$, and let $K \subseteq \mathcal{I} \setminus 0$ denote the set of agents that receive a non-empty bundle in allocation S'_i. Then, we have $\sum_{i \in \mathcal{I} \setminus 0} p_i(S'_i) = \sum_{i \in K} v_i(S'_i) - \sum_{i \in K} \pi_i > \pi_0$, which implies $w(K \cup 0) > \sum_{i \in K \cup 0} \pi_i$ and a coalition, K, that is blocked. □

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