Mechanisms and impossibilities for truthful, envy-free allocations*

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Abstract. We study mechanisms for combinatorial auctions that are simultaneously incentive compatible (IC), envy free (EF) and efficient in settings with *capacitated* valuations — a subclass of subadditive valuations introduced by Cohen et al. [4]. Capacitated agents have valuations which are additive up to a publicly known capacity. The main result of Cohen et al. [4] is the assertion that the Vickrey-Clarke-Groves mechanism with Clarke pivot payments is EF (and clearly IC and efficient) in the case of homogeneous capacities. The main open problem raised by Cohen et al. [4] is whether the existence result extends beyond homogeneous capacities. We resolve the open problem, establishing that no mechanism exists that is simultaneously IC, EF and efficient for capacitated agents with heterogeneous capacities. In addition, we establish the existence of IC, EF, and efficient mechanisms in the special cases of capacitated agents with heterogeneous capacities, where (i) there are only two items; or (ii) the individual item values are binary. Finally, we show that the last existence result does not extend to the stronger notion of *Walrasian* mechanisms, i.e. mechanisms whose allocation and payments correspond to a Walrasian equilibrium.

1 Introduction

A combinatorial auction mechanism takes as input agents' valuations for bundles of items and computes an allocation and payment for each agent. Incentive compatibility (IC) and envy freeness (EF) are two desirable properties of combinatorial auction mechanisms. IC ensures that agents cannot gain by misreporting their private information [11], while EF imposes a notion of fairness on the outcome of the auction. Specifically, EF requires that no agent prefers the allocation and payment of another agent to her own [5, 6, 13, 14, 18].

IC is desirable for various reasons. IC mechanisms create incentives for the agents to report their true values, and as a result, the computed allocation may better optimize the objective of the auctioneer. In addition, IC mechanisms are considered fair in the sense that they do not advantage more sophisticated agents. This is, however, a very weak notion of fairness, and it is well known

* A full version of this paper including all proofs is available on the authors' websites.

that IC mechanisms may not adhere to very basic fairness requirements [1]. In particular, IC mechanisms may produce outcomes which are not EF. This may be problematic in certain settings, such as government run spectrum auctions, since the participants, after observing the outcome, may question the fairness of the auction, and perceive others as being favored by the mechanism. Recent experiments show that people place extremely high value on fairness. For example, Rafaeli et al. [17] show that people care about fairness in queues even more than the actual delay they experience. If outcomes are EF, in contrast, then no agent views other agents' outcomes as preferable.

EF outcomes can be thought of as a relaxation of the outcomes of a *Walrasian equilibrium*. In a Walrasian equilibrium, we have item prices such that every agent receives a bundle that maximizes her utility (i.e., valuation for the bundle minus the sum of the prices of the bundle's items), and the market clears (i.e., every unsold item has a price of zero).[3] If a Walrasian equilibrium exists, then the corresponding outcome is efficient [2] and valuations that are *gross-substitutes* (which is a subclass of subadditive valuations) always admit a Walrasian equilribrium [9].

While every Walrasian equilibrium outcome is clearly EF, the other direction does not hold. In contrast to a Walrasian equilibrium outcome, envy free outcomes assign (arbitrary) bundle prices, which may not correspond to item prices. If the allocation and payments of a mechanism correspond to a Walrasian equilibrium outcome, we say that the mechanism is *Walrasian*.

In this paper, we focus on combinatorial auction mechanisms that are simultaneously IC, EF and efficient; i.e, maximize social welfare. We also consider how our results are affected by replacing the EF requirement with the stronger Walrasian requirement. Because we focus on efficient allocations, the problem of finding IC+(EF or Walrasian) mechanisms reduces to finding payment rules which are IC+(EF or Walrasian), except, possibly, for cases where there may be multiple efficient allocations as in Section 4.

Notably, without the additional EF (or Walrasian) requirement, the family of Vickrey-Clarke-Groves (VCG) mechanisms [3, 8] is known to be IC and efficient for arbitrary valuations. Moreover, the classic results of Green and Laffont [7] and Holmstrom [10] prove that for the efficient allocation and valuations that are connected domains (which include the valuations studied in this paper), any IC mechanism is a VCG mechanism. VCG mechanisms allocate according to an efficient allocation, and determine the payment for each agent in a way that reporting one's true valuations is a dominant strategy. VCG mechanisms are essentially a family of payment rules. The most common payment rule is known as the *Clarke pivot* rule, in which an agent's payment is the externality that the agent imposes on the other agents.

Similarly, without the additional IC requirement, Mu'alem [15] shows that the efficient allocation can always be supported by EF payments. In particular,

[3] We differentiate between a Walrasian equilibrium and a Walrasian equilibrium outcome since a Walrasian equilibrium requires specification of item prices while an outcome simply states the bundle and payment of each agent.

an allocation has supporting EF payments iff it is *locally efficient* — a weaker notion than global efficiency. Thus, an EF and efficient mechanism exists for arbitrary valuation functions.

Therefore, every efficient allocation can be supported by IC payments and can also be supported by EF payments. Unfortunately, it is not always the case that the set of IC payment rules shares a non-empty intersection with the set of EF payment rules, i.e., there may not be a payment rule that can simultaneously satisfy IC and EF. Most of the mechanism design literature focuses on mechanisms that are either IC or EF, but not much attention has been given to the combination of both properties.

One exception is the unit demand case, where each agent desires at most one item. Under these preferences, it is known that VCG with Clarke pivot payments is Walrasian [9, 12] (and is, therefore, clearly IC and EF). Another more recent systematic treatment of the problem is the work of Cohen et al. [4] which considers mechanisms that are IC, EF and efficient for various subadditive valuation classes. In particular, Cohen et al. [4] introduce the class of *capacitated* valuations, which is a natural generalization of unit-demand. Agents with capacitated valuations are associated with a publicly known capacity c and values for individual items. An agent's value for a bundle of items is the sum of the values for the c most valued items in the bundle. We refer to the case where all agents are capacitated and have the same capacity as *homogeneous capacities* and the general case where agents may have arbitrary capacities as the *heterogeneous capacities* case. Because the capacities are publicly known, these classes of valuations are connected and any IC and efficient mechanism must be a VCG mechanism. The results of Cohen et al. [4] are summarized in Figure 1.

	capacitated - heterogeneous	capacitated - homogeneous
IC + Walrasian	**NO [derived by right column]**	**NO** [Cohen et al. [4]] **NO** for binary valuations [**new**]
IC + EF	**NO [new: main result]** **YES** for $n = 2$ [Cohen et al. [4]] **YES** for $m = 2$ [**new**] **YES** for binary valuations [**new**]	**YES** [Cohen et al. [4]]

Fig. 1: This table specifies the existence of a particular type of mechanism (rows) for various families of valuation functions (columns). Efficiency is required in all entries. The results are divided between those that are established by Cohen et al. [4] and those that are established here, indicated as [**new**].

The main result is that the VCG mechanism with Clarke pivot payments is EF for homogeneous capacities. For the broader class of heterogeneous capacities, Cohen et al. [4] show that the VCG mechanism with Clarke pivot payments is not EF, but it is left open whether there exists any mechanism that is simultaneously IC, EF, and efficient. This problem is the main open problem raised by Cohen et al. [4]. For the special case in which there are only two agents (with heterogeneous capacities), it is shown that a particular VCG mechanism (that does not use Clarke pivot payments) is always EF. They also show that under the additional requirement of *no positive transfers* (i.e., payments are weakly positive), no IC, EF, and efficient mechanism exists, even for two agents and two items.

In this paper, we resolve open problems raised in Cohen et al. [4], and establish several additional results for additional natural special cases. Our results are summarized in Figure 1, marked by [**new**]. Our main results are:

- We prove that for heterogeneous capacities, there is no mechanism that is IC, EF and efficient, even if no other requirement (such as no positive transfers) is imposed. To establish this impossibility, we take a computational approach which frames the problem of finding satisfactory VCG payments as a linear program. This result shows that homogeneous capacities is a maximal class that admits an IC, EF, and efficient mechanism. If the capacities are not homogeneous, then IC, EF, and efficient mechanisms no longer exist.
- We devise an IC, EF, and efficient mechanism for heterogeneous capacities in the special case of two items. This result complements the positive result of Cohen et al. [4] which establishes existence for the special case of two agents. Interestingly, the Clarke pivot payment is not EF in either of these cases. Moreover, the two cases rely on different payment rules.
- We then restrict attention to the interesting special case in which agents' valuations for individual items are binary; i.e., in $\{0, 1\}$. We refer to this class as the *binary valuations* class. This is a natural setting where each agent likes a subset of the items but still has a capacity. In this case, there exists a mechanism for heterogeneous capacities that is simultaneously IC, EF, and efficient. In particular, we show that that VCG with Clarke pivot payments is EF if ties in the efficient allocation are broken based on a lexicographic order that favors higher-capacity agents. The tie breaking method is shown to be critical; VCG with Clarke pivot payments is not EF if ties are broken arbitrarily (see Section 4). The proof involves viewing allocations as flows on a particular graph and using augmenting paths and flow decomposition. Similar techniques were used to prove the main result of Cohen et al. [4].
- Finally, we consider mechanisms that are IC, Walrasian, and efficient. We find that, while IC, EF and efficient mechanisms exist for binary valuations and heterogeneous capacities, this result does not extend to IC, Walrasian, and efficienct mechanisms. In particular, we show that there is no IC, Walrasian, and efficient mechanism even for binary valuations and homogeneous capacities.

2 Model and Preliminaries

Suppose we have a set $N = \{1, \ldots, n\}$ of agents and a set $G = \{1, \ldots, m\}$ of goods. We will index agents by i and j and goods by k. Each agent i is associated with a valuation function $v_i : 2^G \to \mathbb{R}_{\geq 0}$ that maps each bundle of goods to the agent's value for that bundle. A valuation profile $v = (v_1, \ldots, v_n)$ consists of a valuation function for each agent. We will often adopt the view of agent i and write a valuation profile as (v_i, v_{-i}), where v_{-i} denotes the valuations of all agents other than i. An allocation $a \in \mathcal{A}$ assigns a bundle of goods to each agent such that no good is given to more than one agent. Let a_i denote the bundle of items allocated to agent i under allocation a. We use the shorthand $v(a)$ to denote the *social welfare* of allocation a, i.e. $\sum_{i=1}^{n} v_i(a_i)$. An allocation is *efficient* if it maximizes social welfare amongst all allocations.

An allocation rule g maps a valuation profile to an allocation, and a payment rule p maps a valuation profile to a payment for each agent, with $g_i(v)$ and $p_i(v)$ denoting the bundle and payment of agent i, respectively. We assume quasi-linear utilities, i.e., the utility of agent i who receives bundle a_i and pays p_i is $v_i(a_i) - p_i$. A mechanism $M = (g, p)$ consists of an allocation rule and payment rule. The following properties of mechanisms are central to our study.

Definition 1. *A mechanism (g, p) is* efficient *if $g(v)$ is an efficient allocation for all v.*

Definition 2. *A mechanism (g, p) is* incentive-compatible *(IC) if there is no benefit to mis-reporting, i.e., for every agent i and every valuation profile (v_i, v_{-i}), $v_i(g_i(v_i, v_{-i})) - p_i(v_i, v_{-i}) \geq v_i(g_i(v'_i, v_{-i})) - p_i(v'_i, v_{-i})$.*

Definition 3. *A mechanism (g, p) is* envy-free *(EF) if no agent prefers the allocation and payment of another agent to her own, i.e., for every i, for every (v_i, v_{-i}), for every $j \neq i$, $v_i(g_i(v_i, v_{-i})) - p_i(v_i, v_{-i}) \geq v_i(g_j(v_i, v_{-i})) - p_j(v_i, v_{-i})$.*

Definition 4. *A mechanism (g, p) is* Walrasian *if the allocation and payments correspond to a Walrasian equilibrium outcome. In other words, there exists a price vector $(q_1, \ldots, q_m)$ such that:*

$$g_i(v) \in \arg\max_{S \subseteq G} \left(v_i(S) - \sum_{k \in S} q_k \right) \tag{1}$$

$$p_i(v) = \sum_{k \in g_i(v)} q_k \tag{2}$$

$$q_k = 0 \quad \textit{if } k \textit{ is unallocated in } g(v) \tag{3}$$

It is easy to verify that a Walrasian mechanism is also EF due to the first condition of Walrasian equilibrium, which stipulates that agents are allocated bundles which maximize their utility given the Walrasian item prices.

In this paper, we study mechanisms where g is an efficient allocation rule. Because we will be considering efficient allocations, it is convenient to introduce the following notation. Given a valuation profile v, Opt refers to an efficient allocation when all agents are considered. There may be multiple efficient allocations due to ties, but we point out where this distinction is important (e.g., in Section 4). Elsewhere, we assume that Opt is any efficient allocation. Opt^{-i} refers to an efficient allocation when agent i is excluded. Since Opt and Opt^{-i} are allocations, Opt_j and Opt_j^{-i} give the allocation of agent j in these allocations.

2.1 Characterization of IC and EF mechanisms

When g is an efficient allocation, IC mechanisms are guaranteed to exist. In particular, Vickrey-Clarke-Groves mechanisms are IC.

Definition 5. *A* Vickrey-Clarke-Groves (VCG) mechanism *is a mechanism* (g, p)*, where* $g(v)$ *is an efficient allocation and* $p(v)$ *takes on the following form,*

$$p_i(v) = h_i(v_{-i}) - \sum_{j \neq i} v_j(Opt_j),$$

where h_i *can be any function of* v_{-i}*.*

One of the most common choices of the h_i function is the *Clarke pivot* payment rule, given by

$$h_i(v_{-i}) = \sum_{j \neq i} v_j(\mathrm{Opt}_j^{-i}). \tag{4}$$

The obtained payment is then $p_i(v) = \sum_{j \neq i} v_j(\mathrm{Opt}_j^{-i}) - \sum_{j \neq i} v_j(\mathrm{Opt}_j)$, which can be interpreted as the externality that agent i imposes on the other agents.

It is well known that VCG mechanisms are IC from the classic results of Clarke [3] and Groves [8]. When the possible valuations of each agent form a connected domain (i.e., there is a path between any two possible valuations that stays within the set of possible valuations), VCG mechanisms are the only IC and efficient mechanisms [7, 10]. Therefore, when considering IC and efficient mechanisms for connected domains, the only flexibility one has is in the choice of the function $h_i(v_{-i})$.

If we consider VCG mechanisms, EF is equivalent to imposing a simple condition on the $h_i(v_{-i})$ functions. When clear in the context, we will often drop the input v_{-i} and simply refer to $h_i(v_{-i})$ using h_i.

Theorem 1. *[16] A VCG mechanism with efficient allocation Opt is EF iff for every valuation profile* v *and for every pair of agents* i, j*:*

$$h_i(v_{-i}) - h_j(v_{-j}) \leq v_j(Opt_j) - v_i(Opt_j). \tag{5}$$

Note that if there are multiple efficient allocations, then EF may depend on which efficient allocations are chosen by the mechanism. This turns out to be the case when we study binary valuations in Section 4. When the choice of efficient allocations is unimportant or when the efficient allocations are unique, the problem of finding IC, EF, and efficient mechanisms for connected domains reduces to finding h_i functions which satisfy (5).

2.2 Restricted classes of valuations

Following Cohen et al. [4], we consider the following classes of valuations. A valuation function is *superadditive* if for any sets $S, T \subseteq G$, $v_i(S) + v_i(T) \leq v_i(S \cup T)$. A valuation function v_i is *subadditive* if for any sets $S, T \subseteq G$, $v_i(S) +$

$v_i(T) \geq v_i(S \cup T)$. Pápai [16] proves that if valuations are superadditive, then VCG with Clarke pivot payments is EF (and trivially IC and efficient). In this paper, we focus on a subset of *subadditive* valuations. A valuation function is *capacitated* with capacity c if it is additive over items up to the capacity c. For sets of items with cardinality greater than c, the value is the sum of the c most valued items. In other words, if we let $top(v_i, S)$ denote the c most valued items in S with $top(v_i, S) = S$ if $|S| \leq c$, then

$$v_i(S) = \sum_{k \in top(v_i, S)} v_i(\{k\})$$

We refer to the case where all agents have the same capacity as the *homogeneous capacities* case, and the more general where capacities can differ as the *heterogeneous capacities* case. We assume that agent capacities are publicly known so that our valuations form a connected domain and VCG mechanisms are the only IC mechanisms.

3 General capacitated valuations

Cohen et al. [4] provide VCG payment rules which are EF for case of two capacitated agents and any number of items. We devise a mechanism for the complementary case, where there are two items and any number of capacitated agents. We also provide a negative result that shows that it is not possible to move beyond these special cases.

Theorem 2. *There exists an IC, EF, and efficient mechanism for two items and any number of capacitated agents.*

Theorem 3. *For capacitated valuations, where the number of items and the number of agents are both at least 3, there is no mechanism that is IC, EF, and efficient.*

The valuations in the proof of Theorem 3 involve agents with capacities 1 and 2, so it is not possible to further generalize the positive result for two items to any number of items but restricted capacities.

4 Binary preferences

Up until now we assumed that agents' valuations for individual items are real numbers. In many real-life settings, however, bidders' preference structure is much simpler. In particular, consider a case where every agent has a set of *desired items*, which are items she is interested in getting. For example, a traveler who needs to express her preferred seats in an airplane would usually have in mind a set of desired seats (e.g., aisle seats). Such a preference structure can be represented by *binary valuations*, where an agent's valuation for every item is either 0 or 1. Moreover, in many situations agents simply do not know their valuations for items. In such cases, the binary valuation structure may serve as a good model, since agents, even if they cannot calculate their exact value for various items, can usually tell whether or not they want some item.

These examples motivate the study of IC, EF, and efficient mechanisms under this restricted preference structure. In particular, we ask whether the impossibility result from the previous section can be circumvented by considering the class of binary valuations (still under capacitated agents). This question is answered in the affirmative. Interestingly, in this case ties among efficient allocations cannot be broken arbitrarily. Only by breaking ties in a very certain way (which we will specify soon) can the desired result be achieved.

The last positive result, however, does not extend to IC, Walrasian, and efficient mechanisms, as even in the more restricted setting — that of agents with homogeneous capacities — there are simple examples that admit no IC, Walrasian, and efficient mechanism.

Theorem 4. *For capacitated agents, where $v_i(\{k\}) \in \{0, 1\}$ for every i, k, there exists an IC, EF, and efficient mechanism.*

Before proceeding with the proof of Theorem 4, we establish some concepts and propositions that are needed in the proof. It will be useful to have in mind the following simple example.

Example 1. Suppose there are three agents, with agents 1 and 2 having capacity 1 and agent 3 having capacity 2. Agents 1 and 3 desire items b, c while agent 2 desires item a.

Because agent values are either 0 or 1, there may be many efficient allocations, and the particular efficient allocation chosen affects the envy-freeness of the resulting mechanism. We consider a *lexicographically-maximal* efficient allocation, where the sorting is done based on the agents' capacities. First, order the agents in a non-increasing order of capacities, arbitrarily breaking ties among agents with the same capacity. Next, compute an efficient allocation that is lexicographically-maximal (among all efficient allocations), according to the order above. i.e., find an efficient allocation such that there is no other efficient allocation that gives an agent with a lower index (i.e. higher capacity) greater value. We only consider allocations in which no agent receives more items than her capacity. This aids in obtaining EF yet is without loss with respect to efficiency because giving an agent more items than her capacity cannot increase welfare. In example 1, a lexicographically-maximal allocation gives agent 3 priority over agents 1 and 2 (since agent 3 has higher capacity). As a result, any lexicographically-maximal efficient allocation must give b, c to agent 3 and a to agent 2.

We show that a lexicographically-maximal efficient allocation, when combined with the Clarke-pivot rule, is IC and EF. Theorem 3.2 from Cohen et al. [4] shows that Clarke-pivot, when used with any efficient allocation, yields a payment rule where agents with higher capacity do not envy agents with lower

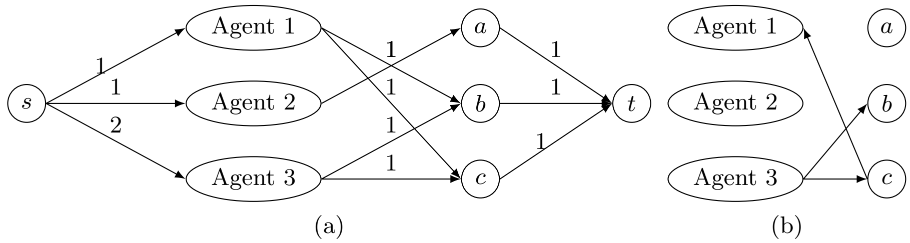

Fig. 2: (a) The graph $G(v)$ for the valuations in Example 1. (b) A graph representing the differences between Opt and D^{-3} for Example 1 (used in the proof of Theorem 4). Edges from agents to items indicate items an agent receives in Opt but not in D^{-3}. Edges from items to agents indicate items an agent receives in D^{-3} but not Opt. Here we assume that Opt allocates a to agent 2 and b, c to agent 3 while D^{-3} allocates a to agent 2 and b to agent 1.

capacity. As a result, to prove that our mechanism is EF, it remains to show that under a lexicographically-maximal efficient allocation and Clarke-pivot, agents with lower capacity do not envy agents with higher capacity.

For a given instance of valuations v, it will be useful to consider a directed graph $G(v)$ similar to Cohen et al. [4]. $G(v)$ contains a source, a node for each agent, a node for each item, and a sink. If an agent desires an item, $G(v)$ contains a directed edge from the agent to the item with capacity 1 (note not to confuse edge capacities in the graph representation with agents' capacities). The source is connected to each agent with a directed edge with capacity equal to the agent's capacity. Each item is connected to the sink with capacity 1. Figure 2(a) depicts this graph for Example 1. An allocation then corresponds to a feasible flow in $G(v)$ by connecting each agent to the items it is allocated and appropriately saturating the edges from the source to the agents and the items to the sink. Any integral flow also corresponds naturally to a feasible allocation.

Consider agents i and j, with agent i having strictly lower capacity than agent j. We wish to show that agent i will not envy agent j. A sufficient condition for this is $h_i - h_j \leq v_j(\mathrm{Opt}_j) - v_i(\mathrm{Opt}_j)$. In the remainder of this section, Opt refers to a lexicographically-maximal efficient allocation and Opt^{-i} refers to a lexicographically-maximal efficient allocation that excludes agent i. Consider the following procedure. Start with the lexicographically-maximal efficient allocation Opt. Remove agent i from this allocation by deallocating agent i (make all of the items allocated to agent i available). Call this allocation C^{-i}. C^{-i} necessarily has weakly less welfare than Opt^{-i} as it is a feasible allocation to the agents other than i. Consider $G(v_{-i})$, the directed graph that excludes agent i, and the flow on $G(v_{-i})$ corresponding to C^{-i}. We can find an allocation D^{-i} with $v(D^{-i}) = v(\mathrm{Opt}^{-i})$ by adding augmenting paths to the flow on $G(v_{-i})$ corresponding to C^{-i}. Since all edge capacities are integer, it is without loss of generality to consider augmenting paths with net flow of 1. It is also without loss of generality to assume that each augmenting path only visits the sink once since any path that visits the sink multiple times contains a smaller augmenting path which visits the sink only once. The following propositions establish properties of these augmenting paths.

Proposition 1. *After each augmenting path, the total set of allocated items increases by exactly one item.*

Proposition 2. *The second to last node (i.e., the node prior to the sink) in each augmenting path is one of the items agent i was originally allocated in Opt.*

Proposition 3. *After adding an augmenting path, every agent other than i receives at least as many items as it did in Opt. Additionally, agent j will receive the same number of items as it did in Opt.*

We are now ready to prove Theorem 4.

Proof. Let Opt be a lexicographically-maximal efficient allocation, and let D^{-i} be the allocation formed by removing agent i and then adding augmenting paths to $G(v_{-i})$. Consider the following bipartite graph G_f and corresponding flow f that relates Opt and D^{-i}. The left hand side has nodes representing agents, and the right hand side has nodes representing items. There is an edge from an agent node to an item node if the agent receives the item in Opt but not in D^{-i}. There is an edge from an item node to an agent if the agent receives the item in D^{-i} but not in Opt. Let there be a flow of 1 on each edge in this graph. Figure 2(b) illustrates G_f and f for Example 1.

Proposition 3 establishes that the only source (node with greater outflow than inflow) is agent i, and that agent j has equal indegree and outdegree since it receives the same number of items in Opt and D^{-i}. Using flow decomposition, we can decompose f into paths and cycles. Each of the paths starts at agent i, with one path for each item agent i was allocated in Opt. By executing a path or cycle, we mean that for every agent to item edge we modify the current allocation by giving the item to the agent, and for every item to agent edge, we remove the item from the agent.

We now construct allocation E^{-j}, which will not allocate any items to agent j, starting from allocation D^{-i}. The items j receives in D^{-i} can be split into two sets. The first set consists of items it also received in Opt, and the second set consists of items it did not receive in Opt. Items in the second set will show up as an item to agent edge in G_f. The sum of the number of items in these two sets will be $v_j(\mathrm{Opt}_j)$ (Proposition 3). For every item given to agent j in both Opt and D^{-i}, give the item to agent i. The remaining items that agent j receives in D^{-i} are part of either a cycle or a path in the flow decomposition of f. For every cycle that contains agent j, execute the cycle, and give the item agent j receives to agent i. This results in agent i receiving some item in Opt_j. For every path that contains agent j, execute the path, stopping at agent j. This results in agent i receiving an item that it desires.

After this process, every agent other than i, j receives the same exact number of items as in D^{-i}. Agent i receives $v_j(\mathrm{Opt}_j)$ items, some of which are in Opt_j

and possibly undesired by agent i (the items j received in both Opt and D^{-i} and the items that were a part of cycles including agent j) and others which are desired by agent i (the items that were part of the paths starting with agent i and ending in agent j). Therefore, agent i receives a bundle that is Opt_j, with some items replaced by items the agent surely desires. As a result, $v_i(E^{-j}) \geq v_i(\mathrm{Opt}_j)$. To complete the proof, we note that $v(E^{-j})$ is a lower bound on $v(\mathrm{Opt}^{-j}) = h_j$ and verify the EF condition for agent i.

Example 1 demonstrates that the tie-breaking rule among efficient allocations is crucial, as some choices of efficient allocations do not yield EF Clarke pivot payments. The restriction to values in $\{0, 1\}$ is tight in sense that if agents have values in $\{r, s\}$ with $r, s > 0$, then VCG with Clarke pivot and lexicographically maximal allocations may no longer be EF. Our final result examines whether this positive result can be extended beyond EF to the stronger notion of Walrasian mechanisms. Notably, for the class of *unit-demand* valuations (homogeneously capacitated agents with capacity 1), VCG with Clarke pivot payments is Walrasian (even for real valuations) [9, 12]. We find that these results cannot be extended, even if we consider homogeneous capacities and binary valuations.

Theorem 5. *There exists no IC, Walrasian, and efficient mechanism for the class of homogeneously capacitated, binary valuations.*

5 Discussion and Open Problems

This work settles the main open question posed by Cohen et al. [4] regarding the existence of an IC, EF and efficient mechanism for valuation classes beyond homogeneous capacities. While there always exists an efficient IC mechanism, and similarly an efficient EF mechanism, there exists no mechanism that simultaneously satisfies both requirements when agents' capacities are heterogeneous. This result eliminates the hope for the existence of IC and EF mechanisms in the more general classes of submodular or subadditive valuations. The impossibility result is accompanied by two positive results, showing that existence of an IC and EF mechanism can be restored if either agents' valuations for individual items are binary or if there are only two items. The former result, however, does not extend to the stronger notion of a Walrasian mechanism, even if valuations are capacitated and binary. The natural future direction, given the impossibility result, is to resort to near-optimal outcomes. What is the best approximation to social welfare that can be achieved by a mechanism that is simultaneously EF and IC, for different valuation classes?

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