

Contingent Payment Mechanisms to Maximize Resource Utilization *

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Abstract

We study the problem of assigning resources in order to maximize the probability of resource utilization. This is a sensible design goal in many settings with shared resources. Each agent has private information about its value distribution for different assignments. A mechanism elicits information about value distributions, using this information to determine an assignment along with payments, including payments that are contingent on utilization. Once assigned, each agent later realizes its value and decides how to act. We seek dominant strategy, no-deficit mechanisms with voluntary participation, that maximize the expected utilization. We adopt two natural design constraints: no-charge to an agent if an assigned resource is utilized, and items should always be assigned. For allocating a single resource, we prove that a *contingent second-price (CSP) mechanism* is unique under these criteria and optimal across a larger class of mechanisms. We extend the mechanism to assign multiple resources, providing theoretical and experimental justification for the performance of the generalized mechanism.

1 Introduction

Consider a member of a gym reserving a spot in a spinning class. When reserving, she may not know whether or not she will be able to attend the class. Consider a city neighborhood allocating time slots for a shared electric vehicle charging station. While reserving, a resident may not know whether she will actually be able to use the station at a particular time. In each case, the assignment decision is associated with an intended action, which is that the resource is utilized by an assigned agent. Attending the spinning class, for example, or using the charging station or the intended (utilization) actions. The problem is to maximize expected utilization in the presence of uncertainty, self interest and private information.

Our focus here is on two period problems, but more general scenarios need not be limited to two periods (e.g. coordinating ride shares for morning commutes, assigning work shifts for members of a childcare co-op, or matching shared bikes with origin-destination pairs). The problem is to choose an assignment in a context where the mechanism can later observe an agent’s utilization action and where payment can be made contingent on this action. We impose a “no-charge” norm that agents should not be charged if they follow the intended action. This appears to fit well many applications, including those of a spinning class provided to members of a club and a public charging station provided to residents of a city.

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We model an agent’s uncertainty about the future as a random variable representing the value of using an assigned resource minus the opportunity cost. Since using the resource may come at the expense of other, more preferred activities or may otherwise not be possible, the random variable may take negative values, modeling the case that an agent prefers an outside option instead of using the resource. The mechanisms that we study first elicit information from agents when each agent knows only the distribution of her value. Based on reports, the mechanisms determine an assignment along with payments, including payments that can be contingent on an agent’s future action.

We seek *dominant strategy*, *no-deficit* and *voluntary participation* mechanisms. In addition, we insist that our mechanisms are *anonymous* (agent symmetric), *deterministic* except when breaking ties, always assign all resources, and satisfy the “no-charge” condition. The design goal is to maximize the expected total utilization subject to these design constraints. simplicity considerations. To motivate the objective, we can consider a social preference that a space in the spinning class be used rather than reserved and wasted, or that ongoing funding to maintain the electric vehicle charging resource relies on it being seen to be used.¹ In both cases, the spinning class and the charging resource can be considered a public good (to gym members or residents, respectively), motivating the no-charge condition.

Before continuing, it is useful to consider the following example in order to understand why the problem is challenging, even with a single resource and even when relaxing some of the design constraints.

Example 1. Consider the problem of allocating a single resource and two agents. Agent 1 values the resource at 10 when she can use it, but this only occurs with probability 0.2 (otherwise her value is -100). Agent 2 values the time slot at 2 when she can use it, and this occurs with probability 0.8 (otherwise her value is -100). Consider the following simple mechanisms:

- Random assignment: each agent is allocated with probability 0.5, no payments are collected, resulting in a probability of utilization of $0.5(0.2 + 0.8) = 0.5$.
- Second price auction: agents 1 and 2 bid their expected non-negative values and thus 2 and 1.6, respectively (reflecting that they will choose not to use the resource if their realized value is negative). Agent 1 is allocated, resulting in a probability of utilization of 0.2. (The second price auction is also precluded because the agent pays 1.6 even when utilizing the resource.)
- Assigning at random, and charging a penalty if the resource is left unused, for example, 20. This mechanism fails because no agent would agree to participate.
- Second price auction with a fixed penalty $C = 2$ if the resource is not used. Both agents will choose to pay C instead of using the resource if the realized value is -100 . Agent 1 bids her expected utility which is for this case $10 \times 0.2 - 2 \times 0.8 = 0.4$. Similarly agent 2 bids $2 \times 0.8 - 2 \times 0.2 = 1.2$ thus agent 2 gets allocated and utilization is 0.8. Note, however, choosing a proper penalty level is not easy: if C is too small ($C < \frac{2}{3}$) agent 1 is still allocated, and if C is too big ($C > 8$) both agents stop participating.

¹Our problem is not the standard one of maximizing the total expected value amongst agents. Leaving aside the no-charge condition, a second-price auction without a penalty for not using the resource would be optimal for this problem. In contrast, our design objective is equivalent to associating a large, societal cost $C > 0$ with an assigned resource being unused, this cost dominating agent values. This is the problem that we solve in this paper. Modifying a second-price auction to sell access along with a fixed penalty C if a resource is not utilized (and allowing for negative bids) optimizes this modified objective but runs at a deficit and ignores the no-charge condition (see Appendix B).

The idea behind the contingent second-price mechanism, which forms the core of our analysis, is to ask agents to bid for the maximum penalty that they are willing to face for not using the resource. In this example, the maximum penalty agent 1 is willing to pay for the right to use the resource is 2.5 (since $10 \times 0.2 - 2.5 \times (1 - 0.2) = 0$) and similarly agent 2 is willing to accept a penalty of \$8. Given this, agent 2 would win the auction and face a penalty of 2.5 and the probability of utilization would be 0.8. Intuitively, the agent who is more likely to show-up will signal this through being willing to face a higher penalty.

In this simple example, the role of mechanism design is simply to address the problem of adverse selection: allocating the agent to the most reliable agent. The general problem that we study also includes a special kind of moral hazard problem where uncertainty at period zero and participation constraints make it impossible to charge unbounded penalties, for example, as would be a standard approach when actions are fully observable. Still, contingent payments can be used to improve utilization by changing the utility calculus of an agent assigned the resource. This second aspect of our problem is illustrated in the following example.

Example 2. Consider the same single resource allocation problem as Example 1, where agent 2 replaced with agent 2' who is happy to use the resource with probability 0.4 (which she values at 4), has a friend coming to visit with probability 0.4 (in which case she values using the resource at -2), and with probability 0.2 cannot use the resource at all (with a very large negative value). The maximum penalty agent 2' is willing to pay for the right to use the resource is 4, in which case she will choose to use the resource to avoid paying 4 when her friend comes to visit, and get $4 \times 0.4 + (-2) \times 0.4 + (-4) \times 0.2 = 0$ in expectation. Thus in the contingent second price mechanism, agent 2' is allocated the resource and faces a penalty of 2.5 (the bid of agent 1). In this case, agent 2' will use the resource with probability 0.8. In contrast, with no penalty, agent 2' will use the resource with probability 0.4.

The general problem that we study is more challenging still, in that we study problems with multiple resources, including multiple, heterogeneous resources.

Our Contributions. For the assignment of a single resource, we study the *contingent second price (CSP) mechanism*, which solicits bids from an agent on the maximum penalty she is willing to make if the resource goes unused, allocates to the highest bidder, and charges the highest bidder the second highest bid only if she does not use the resource. We characterize the mechanisms that satisfy our design criteria for this problem, and prove that the CSP mechanism is the only such mechanism that satisfies these criteria (Theorem 3), achieves optimal utilization amongst a larger class of mechanisms (Theorem 4), and in particular, always has higher expected utilization than the second-price auction.²

In Sections 5, we generalize the CSP mechanism to domains for the assignment of multiple resources, both identical and heterogeneous. For allocating multiple identical resources, we prove that the the contingent $k+1^{\text{th}}$ price mechanism is the only mechanism with the set of desired criteria and is optimal among a larger set of mechanisms (Theorem 5). In lieu of a theoretical result for the case of multiple, heterogeneous resources, we present simulation results in Section 6 and in the appendix, comparing utilizations achieved by generalized-CSP with VCG, first-best and other mechanisms and benchmarks. We show that a significant improvement in utilization is achieved by generalized-CSP, and that allowing reserve prices does not improve utilization on average.

²In the appendix, we provide theoretical and simulation results that on average, a still larger class of mechanisms do not improve expected utilization.

1.1 Related Work

Contingent payments have arisen in previous work on auction design. Prominent examples include auctioning oil drilling licenses [12], royalties [6], and user clicks in ad auctions [18]. In such auctions, payments are contingent on some world state, for example the amount of oil or a click, rather than on bidders' actions. Moreover, the major role of contingent payments in these applications is to improve revenue and hedge risk [16], instead of providing bidders with a way to signal their own, idiosyncratic uncertainty (and thus address moral hazard.)

The timing of information asymmetry plays an important role in principal-agent problems. In the language of contract theory, problems where there is hidden information (e.g. seller's quality [9]) before the time of contracting are usually referred to as *adverse selection*. In contrast, problems for which information asymmetry arises after the time of contracting (e.g. shipping a low quality good) are termed *moral hazard* [11, 13]. The distinction between the two is often blurred in the dynamic settings with multiple periods (see [17, 3]), as is the case for our problem. In our problem, information asymmetries exist both before (value distributions) and after (value realizations) the time of contracting, making the problem quite challenging to solve. Although agents' actions are assumed to be fully observable, uncertainty at period zero and the participation constraints make it impossible to charge unbounded penalties, for example, which is a standard approach when actions are observable in settings with moral hazard.

Courty and Li [8] study the problem of revenue maximization in selling airline tickets, where there is a fixed cost of taking on each passenger and passengers have uncertainty in their values for traveling at the time when tickets are purchased. The optimal mechanisms are proved to be a menu of refund contracts, which can be understood as payments contingent on agents' decisions on whether to take the plane. Although agents' types are modeled as distributions of values in Courty and Li [8], assumptions of first order stochastic dominance or mean-preserving spread are imposed, in which cases the expected utility of a higher type dominates that of a lower type under any contingent payments, making the type space one dimensional. In contrast, we work with general value distributions, and ours is a problem of multi-dimensional mechanism design. Our optimality results also hold profile-by-profile, whereas their results are for a distribution of types.

Other papers study *dynamic mechanism design* in assignment settings, including models with the possibility that workers assigned to tasks will prove to be unreliable [15], and problems where the goal is to maximize expected, discounted value in the presence of uncertainty [14, 5, 2]. We are not aware of any work in the dynamic mechanism design literature that is applicable to our problem, which can be construed as the limit case in which the principal has a very large value (representing society) for agents choosing to utilize the assigned resources.

1.2 Structure of the Paper

Section 2 models agents' types as value distributions and introduces the model of contingent payment mechanisms. The contingent second price (CSP) mechanism is proposed in Section 3, followed by the analysis of dominant strategy equilibrium and analysis of its utilization relative to a second price auction. Characterization and optimality results are presented in Section 4. Section 5 generalizes CSP to the assignment of multiple items, both identical and non-identical, and Section 6 presents simulation results on utilization under various setups.

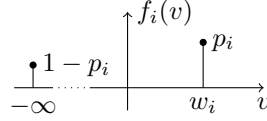


Figure 1: Agent value distribution in the (w_i, p_i) type model.

2 Preliminaries

We first introduce the model for the assignment of a single resource. There is a set $N = \{1, 2, \dots, n\}$ of agents, a single resource, and two time periods. In period zero, the value of agent i for using the resource is uncertain, denoted by a random variable V_i , whose exact value (potentially negative) is not realized until period one. The cumulative distribution function (CDF) F_i on value is agent i 's private information at period zero and corresponds to her *type*.

The assignment is determined in period zero, whereas an allocated agent decides on whether or not to use the resource at period one, after she privately learns the realization of V_i . Define $V_i^+ \triangleq \max\{V_i, 0\}$. We make the following assumptions about F_i :

- (A1) $\mathbb{E}[V_i^+] > 0$, which means V_i takes positive value with non-zero probability. An agent for which this is violated would never be interested in the resource.
- (A2) $\mathbb{E}[V_i^+] < +\infty$, which means agents do not get infinite utility from the resource.
- (A3) $\mathbb{E}[V_i] < 0$, which means the hard commitment of “always use the resource” is not favorable.

It may seem that (A3) is a strong requirement, however, we will see later that an agent for whom (A3) is violated would be willing to commit to pay an unbounded penalty if the resource is not used, which is clearly unreasonable. The following is an example of a distribution with finite support.

Example 3 ((w_i, p_i) model). The value for agent i to use the resource is $w_i > 0$, however, she is able to do so only with probability $p_i \in (0, 1)$. With probability $1 - p_i$, agent i is unable to show up to use the resource due to a hard constraint. The hard constraint can be modeled as V_i taking value $-\infty$ with probability $1 - p_i$:

$$V_i = \begin{cases} w_i, & \text{w.p. } p_i \\ -\infty, & \text{w.p. } 1 - p_i \end{cases}$$

See Figure 1. We have $\mathbb{E}[V_i^+] = w_i p_i > 0$ and $\mathbb{E}[V_i] = -\infty < 0$, and assumptions (A1)-(A3) are satisfied. The $-\infty$ value is not important for (A3). Rather, any value smaller than $-w_i p_i / (1 - p_i)$ would suffice.

2.1 Contingent Payment Mechanisms

At period zero, each agent makes a report r_i from some set of messages $\mathcal{R}$. Let $r = (r_1, \dots, r_n) \in \mathcal{R}^n$ denote a report profile. Based on the reports, an *allocation rule* $x_i(r) : \mathcal{R}^n \mapsto \{0, 1\}$ allocates the right to use the resource to at most one agent, which we denote as $i^* = i^*(r)$. Each agent is charged $t_i(r)$ in period zero. The allocated agent i^* is also charged $t_{i^*}^{(0)}(r)$ or $t_{i^*}^{(1)}(r)$ at the end of period

one, depending on her action of not using or using the resource, respectively. Putting this together, a mechanism is defined by $\mathcal{M} = (\mathcal{R}, x, t, t^{(0)}, t^{(1)})$.³

The time-line of a contingent payment mechanism is as follows:

Period 0:

- For each i , agent i reports r_i to the mechanism based on knowledge of type F_i .
- The mechanism allocates the resource to agent $i^* = x_i(r)$.
- The mechanism collects $t_i(r)$ from each agent, and determines the contingent payments $t_{i^*}^{(0)}(r)$, $t_{i^*}^{(1)}(r)$ for the allocated agent.

Period 1:

- The value v_{i^*} of the allocated agent is realized.
- The allocated agent decides which action to take.
- A contingent payment is collected from the allocated agent.

The design objective is to maximize in dominant strategy equilibrium, and profile-by-profile, the *utilization*:

$$\max_{\mathcal{M}} \sum_{i \in N} s_i^*(\mathcal{M}, F),$$

where $F = (F_1, \dots, F_n)$ is a type profile and $s_i^*(\mathcal{M}, F)$ is the probability that agent i will use the resource given type profile F and rational decisions in period one, under the dominant strategy equilibrium of mechanism $\mathcal{M}$. We impose additional design constraints of participation, no-deficit, and no-charge while also insisting on mechanisms that are anonymous and deterministic (except in the case of ties).

2.2 Design Constraints

We assume that agents are risk-neutral, expected-utility maximizers with quasi-linear utility functions. An unallocated agent's utility is $u_i(r) = -t_i(r)$. For the allocated agent i^* , the utility for using the resource at period one is $v_{i^*} - t_{i^*}^{(1)}(r)$ and the utility for not using the resource is $t_{i^*}^{(0)}(r)$. The rational decision at period one is to use the resource if and only if $v_{i^*} - t_{i^*}^{(1)}(r) \geq -t_{i^*}^{(0)}(r)$ (breaking ties in favor of using the resource). Thus, the expected utility to the allocated agent is

$$u_{i^*}(r) = \mathbb{E} \left[(V_{i^*} - t_{i^*}^{(1)}(r)) \mathbb{1}\{V_{i^*} \geq (t_{i^*}^{(1)}(r) - t_{i^*}^{(0)}(r))\} \right] - t_{i^*}^{(0)}(r) \mathbb{P} \left[V_{i^*} < (t_{i^*}^{(1)}(r) - t_{i^*}^{(0)}(r)) \right] - t_{i^*}(r), \quad (1)$$

where $\mathbb{1}\{\cdot\}$ is the indicator function.

Let $r_{-i} = (r_1, \dots, r_{i-1}, r_{i+1}, \dots, r_n)$ denote the report profile of all agents except i .

Definition 1 (Dominant strategy equilibrium (DSE)). A mechanism $\mathcal{M}$ has a *dominant strategy equilibrium* if, for every agent i , and for all types F_i satisfying (A1)-(A3), there exists a report r_i^* such that

$$u_i(r_i^*, r_{-i}) \geq u_i(r_i, r_{-i}), \quad \forall r_i, \forall r_{-i}.$$

³It is without loss of generality to drop t and just reassign this payment to increment both $t^{(0)}$ and $t^{(1)}$ since an agent's period one decision or expected utility remain unchanged. But it will be convenient to use all three payment terms for describing mechanisms.

A direct mechanism, for which the message space is the type space, is *dominant strategy incentive compatible* (DSIC) if truthful reporting of one's type is a DSE. Let $r^* = (r_1^*, \dots, r_n^*)$ denote the report profile in a DSE under mechanism $\mathcal{M}$.

Definition 2 (Individual rationality (IR)). A mechanism $\mathcal{M}$ is *individually rational* if, for every agent i , and for all types F_i satisfying (A1)-(A3), her expected utility given that she makes rational decisions in period one (if allocated) is non-negative:

$$u_i(r_i^*, r_{-i}) \geq 0, \quad \forall r_{-i}.$$

Equivalently, we say that a mechanism satisfies voluntary participation if and only if it is IR. Note that IR is profile-by-profile, and based on the expected utility before uncertainty is resolved. It is still possible for an agent to get negative utility at the end of period one. We cannot charge unallocated agents without violating IR, thus $t_i(r) \leq 0$ for $i \neq i^*$.

Definition 3 (No deficit (ND)). A mechanism $\mathcal{M}$ is *no deficit* if, for all possible type profiles F and all report profiles r , the expected revenue (total payment) is non-negative, assuming rational behavior of agents in period one:

$$\text{rev}_{\mathcal{M}}(r) \triangleq \sum_i t_i(r) + t_{i^*}^{(0)}(r) \mathbb{P} \left[V_{i^*} < (t_i^{(1)}(r) - t_i^{(0)}(r)) \right] + t_{i^*}^{(1)}(r) \mathbb{P} \left[V_{i^*} \geq (t_i^{(1)}(r) - t_i^{(0)}(r)) \right] \geq 0.$$

We also use the following properties. A mechanism is *anonymous* if the outcome (assignment, payments) is invariant to permuting the identities of agents. *Deterministic* insists that the assignment and payment rules are not randomized unless there is a tie. *Always-allocate* requires that the resource is always allocated as long as there is at least one agent, thus rules out the use of reserve prices which might result in the resource being unallocated. A mechanism satisfies *no-charge* if the allocated agent does not make any payment to the mechanism if the resource is utilized: $t_{i^*} + t_{i^*}^{(1)} \leq 0$.

3 The CSP Mechanism

In the CSP mechanism, agents bids on the penalty that they would be willing to accept if they are allocated the resource and fail to utilize it. The highest bidding agent is allocated the resource, and pays the second highest bid as the no-show penalty.

Definition 4 (Contingent Second-Price Mechanism (CSP)). The *CSP mechanism* collects a single bid $b_i \geq 0$ from each agent. Let $b = (b_1, \dots, b_n)$ be a bid profile.

- Allocate the resource in period zero to the agent with the highest bid (breaking ties at random): $x_{i^*}(b) = 1$ for $i^* \in \arg \max_{i \in N} b_i$, and $x_i(b) = 0$ for $i \neq i^*$.
- The allocated agent i^* makes payment equal to the second-highest bid in the event that she does not use the resource: $t_{i^*}^{(0)} = \arg \max_{i \neq i^*} b_i$. All other payments are zero: $t_i(b) = 0$ for all i and $t_{i^*}^{(1)}(b) = 0$.

We show in this section that under assumptions (A1)-(A3), there exists a simple dominant strategy equilibrium under which CSP always achieves higher utilization than the DSE of a second price auction. We present results on uniqueness and optimality of CSP in Section 4.

3.1 Dominant Strategy Equilibrium under CSP

In developing our main result, we consider a particular agent and simplify notation, so that V , F , t , $t^{(0)}$, and $t^{(1)}$ denote the value, type, period-zero payment and contingent payments for this agent. Refactoring the payments, let:

$$\begin{aligned} y &\triangleq t + t^{(1)} \\ z &\triangleq t^{(0)} - t^{(1)} \end{aligned}$$

Here, y is the “base payment” that an agent makes if she is allocated and chooses to use the resource in period one, and z is the additional “penalty” that she needs to make if she does not use the resource. We call the pair (z, y) a “two-part payment.” The base payment y is always zero under the CSP mechanism.

A rational agent, after learning the realized value v in period one, decides to use the resource if and only if $v \geq -z$ so as to get utility $v - t - t^{(1)} = v - y$ instead of $-t - t^{(0)} = -z - y$, which is the payment if she chooses not use the resource. The base payment y does not affect an agent’s decision in period one on whether to use the resource, and the utilization is $\mathbb{P}[V \geq -z]$.

The expected utility (1) for being allocated the right to use the resource as a function of z and y can be rewritten as

$$u(z, y) = \mathbb{E}[V \cdot \mathbf{1}\{V \geq -z\}] - z \cdot \mathbb{P}[V < -z] - y. \quad (2)$$

We drop the second variable when $y = 0$ (which is always the case under CSP) and write

$$u(z) \triangleq u(z, 0) = \mathbb{E}[V \cdot \mathbf{1}\{V \geq -z\}] - z \cdot \mathbb{P}[V < -z]. \quad (3)$$

This can be expanded into the following form:

$$u(z) = \mathbb{E}[V^+] + \mathbb{E}[V \cdot \mathbf{1}\{-z \leq V < 0\}] - z \cdot \mathbb{P}[V < -z]. \quad (4)$$

The three parts of (4) can be understood as:

- 1) $\mathbb{E}[V^+]$: the highest possible utility that an agent derives from the resource, by utilizing whenever $v \geq 0$ and paying 0 for no-show.
- 2) $\mathbb{E}[V \cdot \mathbf{1}\{-z \leq V < 0\}]$: the expected loss of utility when the realized value is negative and the resource is utilized only to avoid the payment.
- 3) $z \cdot \mathbb{P}[V < -z]$: the expected amount to pay for not utilizing the resource when the realized value is very low.

Lemma 1 (Properties of expected utility). Under (A2), the expected utility $u(z)$ as a function of the penalty z satisfies:

1. $u(z) = \mathbb{E}[V^+] - \int_0^z F(-v)dv$,
2. $u(0) = \mathbb{E}[V^+]$, and $\lim_{z \rightarrow \infty} u(z) = \mathbb{E}[V]$, and
3. $u(z)$ is continuous, convex and monotonically decreasing w.r.t. z .

Part 1 of Lemma 1 can be derived by applying integration by parts to (3), and the proof of the rest of the lemma is straightforward. We defer the formal proofs to Appendix A. Intuitively, the agent gets the expected positive value $\mathbb{E}[V^+]$ when the penalty $z = 0$. As the penalty z increases to infinity, the agent would end up always using the resource and never pays the penalty thus her expected utility converges to $\mathbb{E}[V]$.

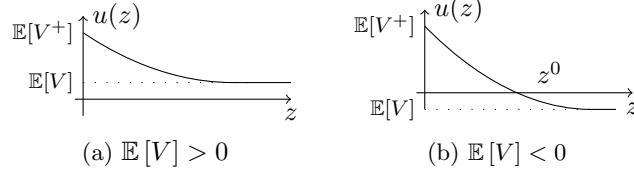


Figure 2: Expected utility for being allocated the resource as a function of penalty z .

Theorem 1 (Dominant Strategy in CSP). *Bidding $b_{\text{CSP}}^* = z^0$, the unique zero-crossing of $u(z)$, is a dominant strategy in CSP given assumptions (A1)-(A3).*

Proof. From parts 2 and 3 of Lemma 1, we know that when $\mathbb{E}[V] < 0$, there is a unique zero crossing z^0 of $u(z)$ s.t. $u(z^0) = 0$, as shown in Figure 2b. In this case, an agent gets non-negative expected utility iff the penalty z is at most z^0 thus z^0 corresponds to the agent's maximum willingness to pay. It is then standard, noting that the bid sets the maximum penalty the agent will face, that bidding z^0 is a dominant strategy. $\square$

Without (A3), we have $u(z) > 0, \forall z \geq 0$, as shown in Figure 2a. This implies that the agent gains in expectation for any penalty z and would always accept such a contract, which we find unreasonable (“pay \$1B if you don’t show up for the spinning class.”) In this case, there is always an incentive to bid higher, and thus no dominant strategy equilibrium exists.

3.2 Better Utilization than Second Price Auction

We prove in this section that CSP always achieves higher utilization than a second price auction (SP).

Lemma 2 (Utilization as slope of $u(z)$). The utilization at penalty z corresponds to the derivative of $u(z)$ w.r.t z :

$$\mathbb{P}[V \geq -z] = 1 - F(-z) = 1 + \frac{d}{dz}u(z),$$

and is monotonically increasing as w.r.t. z .

The proof is straightforward given part 1 of Lemma 1. Intuitively, the agent is more likely to use the resource when the penalty is larger. Moreover, the higher the probability that the resource is to be used at penalty z , the less likely the agent is paying the penalty, thus $u(z)$ decreases slower as z increases, which corresponds to the shallower the slope at z .

As a natural alternative for improving utilization, consider a second-price auction for the right to use the resource, coupled with a fixed penalty $C \geq 0$ if the resource is not used.

Definition 5 (Second-price with fixed penalty C (SP+C)). The *SP+C mechanism* collects a single bid $b_i \geq 0$ from each agent. Let $b = (b_1, \dots, b_n)$ be a bid profile.

- Allocate the resource in period zero to the agent with the highest bid (breaking ties at random): $x_{i^*}(b) = 1$ for $i^* \in \arg \max_{i \in N} b_i$, and $x_i(b) = 0$ for $i \neq i^*$.
- The allocated agent i^* pays the second highest bid $t_{i^*}(b) = \max_{i \neq i^*} b_i$ at period zero, along with the fixed penalty $t_{i^*}^{(0)} = C$ in period one if the resource is not used. All other payments are zero: $t_i(b) = 0$ for $i \neq i^*$ and $t_{i^*}^{(1)}(b) = 0$.

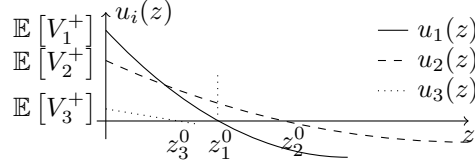


Figure 3: Expected utility for being allocated the resource as a function of penalty z where the SP winner (agent 1) is different from the CSP winner (agent 2).

We know that at period one, the allocated agent will choose to use the resource iff the realized value $v_i^* \geq -C$, thus the expected utility for each agent from the right for using the resource is $\mathbb{E}[V_i \cdot \mathbf{1}\{V_i \geq -C\}] - C \cdot \mathbb{P}[V_i < -C]$, and it is a dominant strategy for agent i to bid

$$b_{i,SP+C}^* = \mathbb{E}[V_i \cdot \mathbf{1}\{V_i \geq -C\}] - C \cdot \mathbb{P}[V_i < -C], \quad (5)$$

if this value is non-negative, and zero otherwise (equivalent to not participating).

The *second price auction* (SP) is attained as a special case, by setting the penalty $C = 0$. In this case, the allocated agent will choose to use the resource as long as $V_i^* \geq 0$, thus it is a dominant strategy for each agent to bid $b_{i,SP}^* = \mathbb{E}[V_i \cdot \mathbf{1}\{V_i \geq 0\}] = \mathbb{E}[V_i^+]$.

Theorem 2. *For any set of agent types satisfying (A1)-(A3), CSP has higher utilization than SP under the dominant strategy equilibria.*

Proof. (Sketch) The full proof is provided in Appendix A. Here we provide the intuition. Recall that in SP, an agent would bid $b_{SP}^* = \mathbb{E}[V^+]$, and an allocated agent would use the resource with probability $\mathbb{P}[V \geq 0]$.

(Case 1) SP and CSP allocate the resource to the same agent, CSP always has (weakly) better utilization since $\mathbb{P}[V \geq -z] \geq \mathbb{P}[V \geq 0]$ for any penalty $z \geq 0$.

(Case 2) SP and CSP allocate the resource to agent 1 and 2 respectively. Ignoring ties, we have $\mathbb{E}[V_1^+] > \mathbb{E}[V_2^+]$, and $z_1^0 < z_2^0$, as shown in Figure 3. In order for this to happen, the slope of $u_2(z)$ at any price $z \in [z_1^0, z_2^0]$ (where the CSP payment resides) must be shallower than that of $u_1(z)$ at $z = 0$, due to the convexity of the expected utility curves. This translates to CSP having strictly higher utilization than SP, recall part 4 of Lemma 1. $\square$

The following examples illustrate the improvement in utilization from CSP over SP, and show that SP can be arbitrarily worse than CSP.

Example 4. (Double gain in CSP) Consider two agents with value distributions and expected utilities as shown in Figure 4. Compared with agent 2, agent 1 has higher value for the facility but with lower probability as well as a greater probability of a hard constraint that completely prevents her using the resource.

Under SP, $b_{SP,1}^* = 20$, $b_{SP,2}^* = 16$ thus agent 1 is allocated and the utilization is $\mathbb{P}[V_1 \geq 0] = 0.2$. Whereas under CSP $b_{CSP,1}^* = z_1^0 = 30$ and $b_{CSP,2}^* = z_2^0 = 60$. Agent 2 is allocated and charged penalty $z^* = b_{CSP,1}^* = 30$ for no-show, thus the utilization is $\mathbb{P}[V_2 \geq -z^*] = 0.8 > 0.4 = \mathbb{P}[V_2 \geq 0]$. The utilization increases in CSP due to better allocated agent selection *and* since the allocated agent has more incentive to use.

Example 5 (SP arbitrarily worse). Under the (w_i, p_i) model introduced in Example 3, the expected utility for agent i given penalty z is $u_i(z) = w_i p_i - (1 - p_i)z$. It is a dominant strategy to bid $b_{i,SP}^* = \mathbb{E}[V_i^+] = w_i p_i$ under SP and the zero-crossing $b_{i,CSP}^* = z_i^0 = w_i p_i / (1 - p_i)$ under CSP.

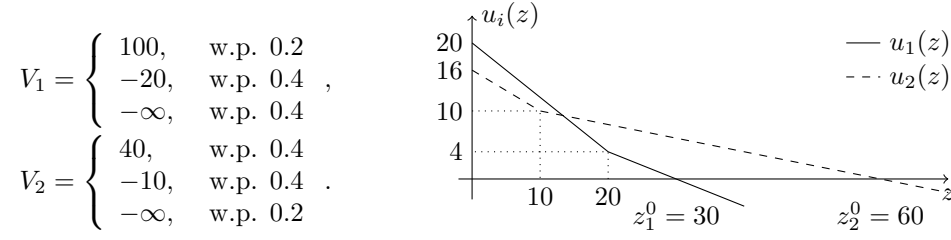


Figure 4: Agents' value distributions and expected utilities in Example 4.

Consider an economy where there are two agents with types: $p_1 = \varepsilon, w_1 = \frac{1}{\varepsilon}$; and $p_2 = 1 - \varepsilon, w_2 = 1$ for some $\varepsilon > 0$ very small. It is easy to verify that agent 1 is allocated under SP since $b_{1,SP} = 1 > b_{2,SP} = 1 - \varepsilon$ whereas agent 2 is allocated under CSP since $b_{2,CSP} = (1 - \varepsilon)/\varepsilon > 1/(1 - \varepsilon) = b_{1,CSP}$. Thus utilization under SP and CSP are ε and $1 - \varepsilon$, respectively; thus, CSP can have arbitrarily better utilization than SP simply by selecting a better winner (the utilization decision of the allocated agent is unaffected by the penalty).

CSP's high utilization comes from two distinct implications of its design:

- Charging a penalty $z > 0$ changes the period one decision of the allocated agent, promoting the resource to be used with higher probability.
- Agents with higher probability of showing-up have utility functions that decrease more slowly with penalty, thus have relatively higher zero-crossing points, bid more in CSP, and are more likely to be allocated.

CSP does not achieve the first-best utilization, though. The third agent for example in Figure 3, who has the shallowest slope and the highest utilization, is not get allocated the resource in CSP.

3.3 Generalizations of CSP

Before continuing, we introduce two natural generalizations of CSP. These enrich the design space for the purpose of our theoretical results in regard to utilization.

The CSP+R Mechanism CSP can be generalized to include a reserve price R , where the resource is only allocated when $b_{i^*} \geq R$ for $i^* \in \arg \max_{i \in N} b_i$, and charge the allocated agent $t_{i^*}^{(0)} = \max(\max_{i \neq i^*} b_i, R)$. We call this the *CSP+R mechanism*.

Under (A1)-(A3), it remains a dominant strategy for agents to bid the zero-crossings of the expected utility curves $b_{i,CSP+R}^* = z_i^0$ under CSP+R. Moreover, we can show that the utilization under CSP+R while setting $R = C$ dominates the utilization of the SP+C mechanism profile-by-profile (see Theorem 6 in Appendix A).

The γ -CSP Mechanism The γ -CSP mechanism allocates the resource to the highest bidder ($x_{i^*}(b) = 1$ for $i^* \in \arg \max_{i \in N} b_i$), charges the second highest bid for no-show ($t_{i^*}^{(0)} = \arg \max_{i \neq i^*} b_i$) and collects an additional γ fraction of this amount when the agent does show up and use the resource: $t_{i^*}^{(1)} = \gamma t_{i^*}^{(0)}$.

SP ($\gamma = 1$) and CSP ($\gamma = 0$) are both special cases of the γ -CSP mechanisms. We can show that under (A1) and (A2), there exists a dominant strategy under the γ -CSP mechanisms for any

$\gamma > 0$, and that the utilization monotonically increases as γ decreases. A more detailed discussion is provided in Section 4 as well as the appendix.

4 Characterization and Optimality of CSP

Consider the following set of properties:

- | | |
|-----------------------------------|--|
| P1. Dominant-strategy equilibrium | P5. Deterministic (unless breaking ties) |
| P2. Individually rational | P6. Always-allocate (“no waste”) |
| P3. No deficit | P7. No positive payment if resource is used: |
| P4. Anonymous | $t_{i^*} + t_{i^*}^{(1)} \leq 0$ (“no charge”) |

In this section, we characterize the possible outcomes of any mechanism under (P1)-(P5), when the type space is the set of all value distributions satisfying (A1)-(A3), and prove the following uniqueness result:

Theorem 3. *Assume the type space is the set of all value distributions satisfying (A1)-(A3). CSP is the only mechanism with properties (P1)-(P7).*

In addition, we prove that:

- (i) CSP achieves optimal utilization within certain type classes,
- (ii) CSP is provably better than a broad subclass of mechanisms satisfying (P1)-(P6), including SP and γ -CSP, and
- (iii) Relaxing (P6) by adding reserve prices rarely improves utilization.

4.1 Characterization under (P1)-(P5)

We first characterize all possible outcomes for mechanisms satisfying (P1)-(P5). Readers who are more interested in utilization and optimality of CSP can also skip directly to Section 4.2 where we compare CSP with other mechanisms.

For the formal analysis, we appeal to the revelation principle, and consider the space of direct-revelation mechanisms (DRMs). We first introduce some additional terminology.

Zero-profit Curves Recall that in the CSP mechanism the base payment $y = t + t^{(1)}$ is always zero. To study general mechanisms, we move back to the full (z, y) payment space, and work with *iso-profit curves* which are sets of (z, y) pairs for which $u(z, y) = c$ for some constant c . See Figure 5a.

Because $u(z, y) = u(z) - y$ (see (2) and (3)), the *zero-profit curve* (i.e. where $c = 0$, the solid line depicted in Figure 5a) is characterized by $\{(z, y) \mid y = u(z)\}$. This has the same shape as the $u(z)$ curve in Figure 2 and is continuously decreasing and convex according to Lemma 1. Observing $\frac{\partial}{\partial y}u(z, y) = 1$ and $\frac{\partial}{\partial z}u(z, y) = \frac{d}{dz}u(z)$, we know that other iso-profit curves are vertical shifts of the zero-profit curve, and that the utilization for an agent facing payments (z, y) is still $\mathbb{P}[V \geq -z] = 1 + \frac{d}{dz}u(z)$, which directly relates to the slope of the zero-profit curve at point (z, y) .

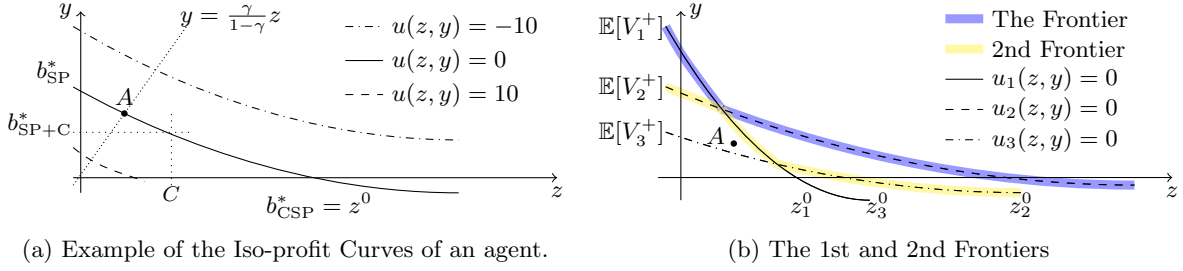


Figure 5: Illustration of iso-profit curves and the frontiers

DSE Bids Since the expected utility $u(z, y)$ decreases in z and y , an agent gets non-negative expected utility iff the two-part payment she faces is weakly below the zero-profit curve, which means that the zero-profit curve characterizes an agent's maximal willingness to pay. This is helpful for the analysis of the dominant strategies under various mechanisms.

Under CSP, we have $y = 0$ and it is a dominant strategy for an agent to bid the crossing point of $y = 0$ and the agent's zero-profit curve (Theorem 1). Similarly, we can check that in SP and SP+C, the DSE bids (5) correspond to the y components of the crossing points of the zero-profit curve with the $z = 0$, $z = C$ lines (see Figure 5a). Under γ -CSP, the dominant strategy is to bid the sum of the two coordinates of point A ($z_A + y_A$), which is the crossing point between the zero-profit curve and $y = \frac{\gamma}{1-\gamma}$, the subspace of payments under the γ -CSP mechanism (see Theorem 7 in Appendix A.)

The Frontier Define the *frontier* of an economy to be the upper-envelope of the zero-profit curves of each agent, i.e., $\{(z, y) \mid y = \bar{u}_N(z)\}$ where $\bar{u}_N(z) \triangleq \max_{i \in N} u_i(z)$. For any penalty z , this is the maximum base payment any agent is willing to pay. $\bar{u}_N(z)$ is continuously decreasing and convex, with $\frac{d}{dz} \bar{u}_N(z)$ increasing in z . We similarly define the k^{th} frontier of the economy. See Figure 5b.

IR and ND Ranges We call the area weakly below an agent's zero-profit curve the *IR-range* for the agent. Any payment above the zero-profit curve would give the agent negative expected utility. For any two-part payment, the expected revenue is

$$rev(z, y) = y + z \cdot \mathbb{P}[V < -z].$$

We define the set of payments (z, y) for which $rev(z, y)$ is non-negative as the *ND-range* (no-deficit range) for this agent. Under any mechanism that is IR and ND, the payment facing the allocated agent must lie in the intersection of the IR and ND ranges.

Example 6. Consider the simple $w_i p_i$ model (Example 3). The zero-profit curve is given by $y = u(z) = w_i p_i - (1 - p_i)z$, which is the upper bound of the IR range. The expected revenue that the mechanism collects is $rev(z, y) = y + (1 - p_i)z$, thus the ND range is lower-bounded by $y = -(1 - p_i)z$. See Figure 6b(i).

Example 7 (Exponential model). Under the exponential model, the utility for agent i to use the resource is a fixed value $w_i > 0$ minus a random opportunity cost, which is exponentially distributed with parameter $\lambda_i > 0$. The value distribution is

$$f_i(v) = \begin{cases} \lambda_i e^{\lambda_i(v-w_i)}, & v \leq w_i \\ 0, & v > w_i \end{cases},$$

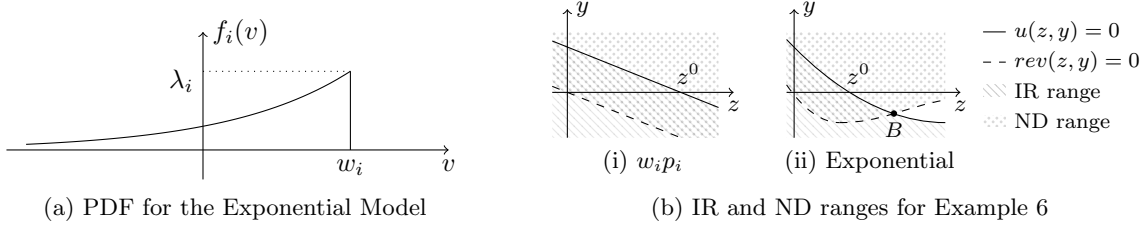


Figure 6: The frontiers and the IR and ND ranges.

as illustrated in Figure 6a. The expectation $\mathbb{E}[V_i] = w_i - 1/\lambda_i$, where $1/\lambda_i$ is the expected value of the opportunity cost. With $w_i < 1/\lambda_i$, assumptions (A1)-(A3) are satisfied. For an agent with an exponential value distribution, the zero-profit curve, IR and ND ranges are as shown in Figure 6b(ii) (see derivations in Appendix D). We can see that no IR and ND mechanism could set a higher no-show penalty z beyond point B , thus with complete information, the highest possible utilization would be $\mathbb{P}[V \geq -z_B]$.

We now state and prove the technical lemmas for the characterization results in this section.

Lemma 3 (Possible Allocation and Payments). For any mechanism that satisfies (P1), (P2), (P4) and (P5), the zero-profit curve of the allocated agent i^* must be a part of the frontier of all agents, and the two-part payment (z^*, y^*) facing the allocated agent must be weakly above the frontier of the rest of the economy.

Proof. The second part of Lemma 3 requires unallocated agents get non-positive utility at the payment facing the allocated agent: $\forall i \neq i^*, u_i(z^*, y^*) \leq 0$. If this is the case, we know from IR that $u_{i^*}(z^*, y^*) \geq 0 \geq u_i(z^*, y^*)$ thus $i^* \in \arg \max_{i \in N} u_i(z^*)$, i.e. agent i^* must reside on the frontier of the economy at z^* . What is left to show is that the existence of an unallocated agent $i' \neq i^*$ s.t. $u_{i'}(z^*, y^*) > 0$ leads to a contradiction. We prove this for deterministic tie-breaking and no-payment to unallocated agents and leave the full proof in Appendix A.

Assume that such agent i' exists. Consider the economy where the allocated agent i^* is replaced by another agent with the type of agent i' . With deterministic tie-breaking, at least one of the two agents with type $u_{i'}$ would not be allocated, thus would have the incentive to report the type of agent i^* , get allocated and get non-negative utility. As an example, in the economy depicted in Figure 5b, if agent 2 is allocated and charged a payment at point A , then in the economy with agents with types u_1, u_1, u_3 , at least one of the agents with type u_1 is not allocated, thus would benefit from pretending to have type u_2 . This violates DSIC. $\square$

Now we show that no mechanism under (P1)-(P5) can pay agents for “showing up”.

Lemma 4 (No Negative Base Payments). In any mechanism satisfying (P1)-(P5), the base-payment facing the allocated agent $y^* = t_{i^*} + t_{i^*}^{(1)} \geq 0$.

Proof. Assume otherwise, that there is an economy E where the allocated agent (say agent 1) is charged payment (z_1, y_1) where $y_1 < 0$. Consider economy E' where agent 1 is replaced by agent $1'$ who has $w_i p_i$ type where $p_{1'} = 1 + y_1/(2z_1)$. It's easy to check that $u_{1'}(z_1, y_1) = w_{1'} p_{1'} - y_1/2 > 0$ thus agent $1'$ must be allocated and guaranteed utility at least $u_{1'}(z_1, y_1)$. Otherwise she has the incentive to deviate and report the type of agent 1. However, for all (z, y) s.t. $u_{1'}(z, y) \geq u_{1'}(z_1, y_1)$, $rev_{1'}(z, y) = y + (1 - p_{1'})z = w_{1'} p_{1'} - u_{1'}(z, y) < y_1/2 < 0$, thus the mechanism does not satisfy ND. $\square$

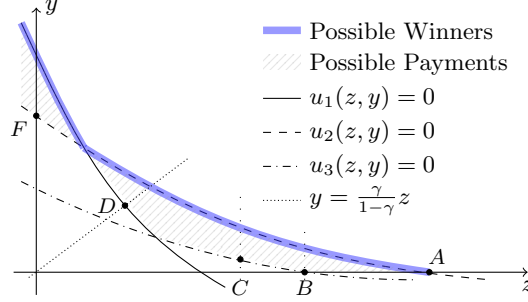


Figure 7: Characterization of possible outcomes for mechanisms under (P1)-(P5).

Lemmas 3 and 4 provided a characterization of all possible outcomes of any mechanism satisfying (P1)-(P5), as shown in Figure 7: payments reside in between the frontier and the second frontier above the horizontal axis, and the resource must be allocated to the agent on the frontier at the point above the payment. Let z_N^0 be the zero-crossing of the frontier $\bar{u}_N(z)$, i.e. the highest zero-crossing of all agents, the following lemma characterizes the utilization achievable by any mechanism satisfying (P1)-(P5).

Lemma 5 (Utilization Characterization). For any mechanism $\mathcal{M}$ satisfying (P1)-(P5), for any economy, when the allocated agent is charged a two-part payment (z^*, y^*) , the utilization corresponds to the slope of the frontier: $ut_M = \frac{d}{dz} \bar{u}_N(z)|_{z=z^*} + 1$, which attains its maximum at $z^* = z_N^0$.

Proof. Since the allocated agent i^* must be on the frontier above the payment (z^*, y^*) , we know that $u_{i^*}(z) = \bar{u}_N(z)$ for z in a small interval around z^* . The utilization of the allocated agent therefore corresponds to the slope of the frontier $\mathbb{P}[V_{i^*} \geq -z^*] = \frac{d}{dz} u_{i^*}(z)|_{z=z^*} + 1 = \frac{d}{dz} \bar{u}_N(z)|_{z=z^*} + 1$. We know from Lemmas 3 and 4 that under (P1)-(P5) the allocated agent can only be charged penalty $z^* \leq z_N^0$, thus $\frac{d}{dz} \bar{u}_N(z) + 1$ for $z \leq z_N^0$ is all possible utilization achievable by any mechanism under (P1)-(P5).

Since $\bar{u}_N(z)$ is convex, $\frac{d}{dz} \bar{u}_N(z)$ is monotonically increasing in z , thus the utilization of any mechanism under (P1)-(P5) is upper bounded by $\frac{d}{dz} \bar{u}_N(z)|_{z=z_N^0} + 1$, which corresponds to allocating the resource to the agent with the highest zero-crossing and charge her the penalty equal to the highest zero-crossing (point A in Figure 7). $\square$

Observe that CSP allocates to the first-best agent but charges a smaller penalty (point B, the second highest zero-crossing), thus achieving a lower utilization.

4.2 Uniqueness and Optimality of CSP

Theorem 3. Assume the type space is the set of all value distributions satisfying (A1)-(A3). CSP is the only mechanism with properties (P1)-(P7).

Proof. From Lemma 4 and the assumption (P7), we know that in any economy, the payment (z^*, y^*) facing the allocated agent i^* must satisfy $y^* = 0$. W.l.o.g assume agents are ordered in decreasing-order in their zero-crossings $z_1^0 \geq z_2^0 \geq \dots \geq z_n^0$. Consider for now the case with no tie at the highest zero-crossings: $z_1^0 > z_2^0$. Lemma 3 implies that agent 1 must be allocated and charged a penalty $z^* \in [z_2^0, z_1^0]$.

We now argue $z^* = z_2^0$ must hold, and thus the CSP outcome. Assume otherwise and $z^* > z_2^0$, consider the economy in which agent 1 is replaced by agent 1' with zero-crossing $z_{1'}^0 > z_2^0$ but

$z_{1'}^0 < z^*$. Agent 1' must be allocated from the above argument and (P6), and that the penalty that she faces z' must be smaller than z^* , otherwise her expected utility $u_{1'}(z', 0) \leq u_{1'}(z^*) < u_{1'}(z_{1'}^0) = 0$ which violates IR. This gives agent 1 in the original economy a useful deviation, which is to report the type of agent 1', getting allocated and charged a smaller penalty.

When there is a tie at the highest zero-crossings, Lemma 3 implies that the only possible outcomes would be one of the agents with the highest zero-crossings getting allocated and payment must be also be the highest zero-crossing $(z_N^0, 0)$ — the only point weakly below the frontier, above the second frontier and with $y = 0$. This is also the CSP outcome.⁴ This completes the proof of the uniqueness of CSP. $\square$

We may also ask what happens once (P6) or (P7) are relaxed. We can show that CSP is not dominated by any other DSIC mechanism satisfying (P1)-(P5) (see Theorem 8 in Appendix A.) For the simple $w_i p_i$ model, where the penalty does not improve the utilization of any agent, CSP remains optimal.

Corollary 1. *Assume the type space is the set of all $w_i p_i$ value distributions. CSP achieves highest possible utilization among mechanisms satisfying (P1)-(P5), type profile by type profile.*

The proof is straightforward. Lemma 5 proves that under (P1)-(P5), a mechanism can only achieve higher utilization than that of CSP by allocating to the same allocated agent (the CSP winner), and charge a penalty higher than the second largest zero-crossing. But this does not improve utilization for the $w_i p_i$ model.

What about the effect of relaxing (P7) in more general settings? (P7) restricts agents' payments to a one-dimensional space—the $y = 0$ axis. We relax this to allow two-part payments (y, z) , while requiring them to be independent of an agent's report.⁵ An illustrative class is the family of γ -CSP mechanisms, which is rich enough to include both the CSP and the SP mechanisms.

Theorem 4. *For any set of agent types satisfying (A1)-(A3), CSP achieves highest utilization among all mechanisms that satisfy (P1)-(P6) and where the payments are agent-independent.*

Proof. First, Lemma 3 shows that the two-part payment facing any allocated agent must be weakly below the second frontier. We now prove that the payment cannot be strictly above the second frontier. Assume otherwise, that agent 1 is allocated in some economy E and charged a two-part payment (z_1, y_1) that is strictly above the second frontier. Consider the economy E' with agents $1', 2, \dots, n$ where $2, \dots, n$ belong to the original economy E , where the zero-profit curve of agent 1' strictly dominates the frontier of agents $2-n$, but resides below (z_1, y_1) . This is always possible, observing that the frontier of agents $2-n$ is convex, and corresponds to a valid type under (A1)-(A3). We know from Lemma 3 that agent 1' must be allocated in economy E' , since she is the only agent that resides on the frontier, and that the resource must be allocated under (P6). We know from agent-independence that agent 1' must be charged (z_1, y_1) as well, thus IR is violated. The proof of the theorem now follows, observing that payments must be on the second frontier and that the CSP payment has the largest penalty component among all such two-part payments. $\square$

As a corollary of Theorem 4, CSP achieves highest utilization among the larger class of γ -CSP mechanisms.

⁴In case of ties, a direct-revelation version of CSP may break ties in favor of the agent with highest utilization at the zero-crossing, and achieve higher utilization than breaking ties uniformly at random. This does not affect incentives since when tied, both allocated and unallocated agents get expected utility zero.

⁵This is not fully general in our setting because there can, in general, be a set of different contracts available to an agent, with an agent's type determining which contract it is chosen.

Corollary 2. *For any set of agent types satisfying (A1)-(A3), CSP dominates γ -CSP mechanism in utilization.*

We might also relax (P6), and allow mechanisms that do not always allocate. The most natural way to do this is via reserve prices. Assume the zero-crossings of agents are ordered $z_1^0 \geq \dots \geq z_n^0$. We know from the earlier discussion that agents bid the zero-crossings of their expected utility functions under CSP+R, and it follows that CSP+R allocates to agent 1 the CSP winner if $z_1^0 \geq R$, and improves utilization if $R > z_2^0$ since agent 1 is charged a higher penalty. On the other hand, CSP+R leaves the resource unallocated if $z_1^0 < R$.

Simulation results show that reserve prices do not help for improving utilization for simple type distributions (see Section 6). We can also prove that for one agent with exponential or uniform type model, the optimal reserve is zero, i.e. the resource should be given to the agent for free even without charging a fixed post penalty (see Appendix A.1).

This stands in stark contrast to the role of reserve prices for improving the revenue in auctions. The reason is as follows. When an item remains unallocated due to a small reserve price in a first or second price auction, the revenue loss is small (bounded by the reserve price). In contrast, in our problem, the utilization given penalty z consists of two parts:

$$ut_{i^*}(z) = \mathbb{P}[V_{i^*} \geq z] = \mathbb{P}[V_{i^*} \geq 0] + \mathbb{P}[-z \leq V_{i^*} < 0] \quad (6)$$

The first part in (6) corresponds to the probability that the agent would use the resource if she is given the resource free of charge, and the second part is the improvement in utilization due to the penalty z . While reserve prices improve the utilization captured in the second part, the average loss of the first part due to non-allocation is large and thus usually results in an overall negative effect on utilization.

Properties (P2)-(P5), on the other hand, are crucial for our results. Relaxing each one in turn, we provide a mechanism that achieves higher utilization than CSP for some economies.

- (P2) If there is no IR constraint, a mechanism can charge an agent an arbitrarily large penalty s.t. the agent's probability of using the resource is significantly improved.
- (P3) Without budget constraint, we can design a mechanism which pays agents a huge bonus for showing up ($y \rightarrow -\infty$) and charge a huge penalty for no show ($z \rightarrow \infty$.) By choosing (z, y) below an agent's zero-profit curve, we would get very large utilization without violating IR.
- (P5) Consider an agent with (w_i, p_i) type with $w_i = \varepsilon^2$ and $p_i = 1 - \varepsilon$ for some ε very small. Her zero-crossing is very small thus would not get allocated the resource under CSP, even though she would show up with probability almost one. A mechanism that allocates the resource uniformly at random allocates to this agent with non-zero probability, in which case has higher utilization than CSP.
- (P4) Similar to P5, if a mechanism is not anonymous, it can always allocate the resource to the above agent, regardless the reports of the rest of the agents, thus may also have a higher utilization than CSP.

5 Assignment of Multiple Resources

In this section, we generalize the model for allocating multiple resources, and propose two DSIC mechanisms, the contingent $k + 1$ price mechanism for allocating k identical resources, and the

generalized CSP mechanism for allocating k different resources. We assume unit demand, so that each agent is interested in at most one resource.

5.1 k Identical Resources

Consider first the scenario of allocating the k spots in a spinning class, where the multiple resources to be allocated are identical. An agent's type is still F_i , describing its value distribution for any one resource. The CSP mechanism can be generalized as follows:

Definition 6 (Contingent $k + 1^{\text{th}}$ Price Mechanism). The *contingent $k + 1^{\text{th}}$ price mechanism* collects a single bid b_i from each agent. Let $b = (b_1, \dots, b_n)$ be a bid profile and w.l.o.g. assume $b_1 \geq b_2 \geq \dots \geq b_n$.

- Allocate the resource in period zero to the k highest bidders (breaking ties at random): $x_i(b) = 1$ for $i \leq k$ and $x_i(b) = 0$, otherwise.
- The allocated agents pay the penalty equal to the $k + 1^{\text{th}}$ highest bid $t_{i^*}^{(0)}(b) = b_{k+1}$ in the scenario of no-show at the end of period one. All other payments are zero.

We can show that it is still a dominant strategy for agents to bid their zero-crossings z_i^0 under the contingent $k + 1^{\text{th}}$ price mechanism. The CSP+R, SP and SP+C mechanisms can also be generalized in the same way to allocate k identical resources, and the dominant strategies remain the same as in the single resource assignment problem.

Theorem 5. *Assume the type space is the set of all value distributions satisfying (A1)-(A3). For allocating k identical resources, the contingent $k + 1^{\text{th}}$ price mechanism is the only mechanism satisfying (P1)-(P7). Moreover, it achieves highest utilization among all mechanisms that satisfy (P1)-(P6) and where the payments are agent-independent.*

In particular, the contingent $k + 1^{\text{th}}$ price mechanism dominates the generalization of γ -CSP mechanisms for any γ , including the $k + 1^{\text{th}}$ price auction (i.e., the generalization of SP).

The proof of the theorem follows the same scheme as Theorems 3 and 4, after observing that the allocated agents must reside somewhere on the top k upper envelopes of the zero-profit curves of all agents, the payment must reside on the $k + 1^{\text{th}}$ envelope, and that the summation of the slopes of the top k upper envelopes increases monotonically as the penalty increases. We can also show that with a reserve penalty R , the contingent $k + 1^{\text{th}}$ price mechanism is always better than a $k + 1^{\text{th}}$ price auction with fixed penalty $C = R$.

5.2 k Heterogeneous Resources

Consider now the problem of allocating k resources that are not identical, as in the problem of assigning residents to a shared charging station, where there are k different time slots.

Let $N = \{1, 2, \dots, n\}$ be the set of agents and $M = \{a, b, \dots, k\}$ be the set of resources. For each resource $j \in M$, the value for agent i to use the resource $V_{i,j}$ is modeled as a random variable with distribution function $F_{i,j}$. We assume that the values $V_{i,j}$'s are independent, and that the distributions $F_{i,j}$'s satisfy (A1)-(A3).

This is the utilization-maximization version of the classical unit-demand assignment problem. The naïve generalization of CSP, which collects a single bid from agents and charges allocated agents the externality they impose on the rest of the agents computed from the bids, fails to be incentive compatible because the expected utility as a function of the penalty $u_{i,j}(z)$ is not quasi-linear in z .

Algorithm 1: Alaei's mechanism ALG_1

Input: Utility functions $u = \{u_{i,j}(z)\}_{i \in N, j \in M}$
Output: Assignment $x_{i,j} \in \{0, 1\}$, payments z_i and resulting expected utilities π_i
// If there is only one agent, the agent gets her favourite resource free of charge
1 **if** $|N| = 1$ **then**
2 $x_{1,j^*} \leftarrow 1$ for $j^* = \arg \max u_{1,j}(0)$; $x_{1,j} \leftarrow 0$ for $j \neq j^*$; $z_1 \leftarrow 0$; $\pi_1 \leftarrow u_{1,j^*}(0)$.
3
4 **else**
5 // If there is only one resource to allocate, compute the CSP outcome
6 **if** $|M| = 1$ **then**
7 $x_{i^*,1} \leftarrow 1$ for $i^* = \arg \max z_{i,1}^0$ where $z_{i,1}^0 = u_{i,1}^{-1}(0)$; $z_{i^*} \leftarrow \max_{i \neq i^*} z_{i,1}^0$; $\pi_{i^*} \leftarrow u_{i^*,1}(z_{i^*})$;
8 $x_{i,1} \leftarrow 0$, $z_i \leftarrow 0$ and $\pi_i \leftarrow 0$ for all $i \neq i^*$.
9 // Allocating $|M| > 1$ resources to $|N| > 1$ agents
10 **else**
11 **for** $i \in N, j \in M$ **do**
12 // Compute the outcome in the economy without agent i or resource j
13 $(x_{(-i,-j)}, z_{(-i,-j)}, \pi_{(-i,-j)}) \leftarrow \text{ALG}_1(u_{-i,-j})$;
14 **for** $i' \neq i$ **do** $q_{(-i,-j),i'} \leftarrow u_{i',j}^{-1}(\pi_{(-i,-j),i'})$ $z_{i,j} \leftarrow \max\{0, \max_{i' \neq i} q_{(-i,-j),i'}\}$
15 // Compute the agent-maximizing allocation and determine the payments
16 **for** $i \in N$ **do**
17 **if** $\max_{i \in M} u_{i,j}(z_{i,j}) > 0$ **then**
18 $x_{i,j^*} \leftarrow 1$ for $j = \arg \max u_{i,j}(z_{i,j})$; $x_{i,j} \leftarrow 0$ for $j \neq j^*$;
19 $z_i \leftarrow z_{i,j^*}$; $\pi_i \leftarrow u_{i,j^*}(z_i)$;
20 **else** $x_{i,j} \leftarrow 0, \forall j$; $z_i \leftarrow 0$; $\pi_i \leftarrow 0$
21 **return** x, z, π .

We use the algorithm for non-quasi-linear assignment (Alaei et al. [1]) to generalize the CSP mechanism to this setting. The generalized CSP mechanism computes agent-independent prices recursively, and allocates in the agent-maximizing manner.

Definition 7 (Generalized CSP Mechanism). The generalized CSP mechanism (GCSP) collects from each agent the value distribution $F_{i,j}$ for all alternatives, computes the expected utilities as functions of the no-show penalty $u_{i,j}(z)$ according to (4).

- Allocation rule: assign resources to agents according to Algorithm 1.
- Payment rule: charge each allocated agent $t_i^{(0)} = z_i$, where z_i is the payment determined by Algorithm 1. All other payments are zero.

Intuitively, Algorithm 1 computes for each agent i and each resource j the critical payment $z_{i,j}$, at which the rest of the agents $-i$ are indifferent between getting j at this payment $z_{i,j}$ or getting the outcome in the economy without agent i or resource j . By construction, the critical penalties $z_{i,j}$ are independent to agent i 's own reports, and the assignment rule is agent maximizing thus the mechanism Algorithm 1 is DSIC, thus so is GCSP. We can verify that GCSP satisfies (P2)-(P7).

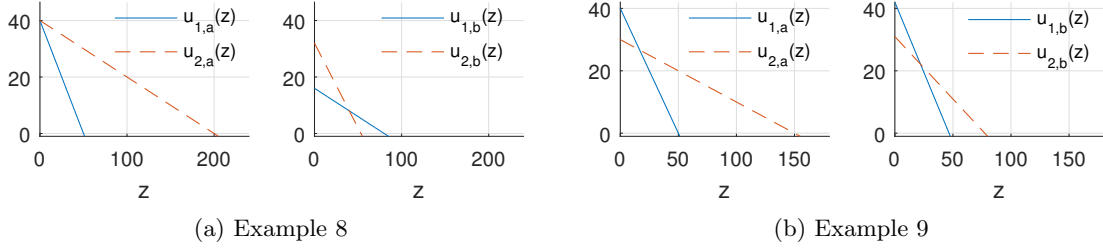


Figure 8: An example of utility curves for the heterogeneous assignment problem.

We can compare this mechanism with the appropriate generalization of the SP mechanism, in this case the Vickrey-Clarke-Groves (VCG) mechanism [19, 7, 10]. In particular, we use VCG with a fixed penalty (termed VCG+C). This collects a single bid from each agent for each alternative $b_{i,j}$, computes the assignment that maximized the summation of total bids, and charges each allocated agent at period zero the negative externality (in terms of bids) that she imposes on the rest of the agents. It is a dominant strategy to bid true values in VCG mechanisms, and with the same analysis as that of the SP+C mechanism, we can show that it is a dominant strategy for each agent to bid $b_{i,j}^* = \mathbb{E}[V_{i,j} \cdot \mathbf{1}\{V_{i,j} \geq -C\}] - C \cdot \mathbb{P}[V_{i,j} < -C]$ on each alternative j if this is non-negative, but we omit the formal definition and the proof.

Unlike for the assignment of identical resources, the utilization of generalized CSP and VCG are incomparable. We show this through a pair of examples. Still, simulations results in Section 6 show that the generalized-CSP mechanism almost always achieves higher utilization than VCG, except for some extreme economies with little competition.

Example 8 (GCSP beats VCG). Consider an economy with the following two agents 1 and 2, two items a and b , and the $w_i p_i$ type model:

- Item a : $v_{1,a} = 200$, $p_{1,a} = 0.2$; $v_{2,a} = 50$, $p_{2,a} = 0.8$.
- Item b : $v_{1,b} = 20$, $p_{1,b} = 0.8$; $v_{2,b} = 80$, $p_{2,b} = 0.4$.

The expected utility curves of the two agents for the two resources are as shown in Figure 8. It's easy to check that agents bid $b_{i,j}^* = w_{i,j} p_{i,j}$, agent 1 is allocated resource a and agent 2 is allocated resource b , and the total utilization is $p_{1,a} + p_{2,b} = 0.6$.

We now compute the outcome under generalized CSP. The expected utility functions are of the form $u_{i,j}(z) = v_{i,j} p_{i,j} - (1 - p_{i,j})z$ for all i, j . We feed u into Algorithm 1, which computes agent-independent prices $z_{i,j}$ for all i, j then do agent-maximizing allocation. Recall that $\pi_{(-i,-j),i'}$ denotes the utility of i' in the economy without i, j . First, for agent 1 and resource a , we need to compute the outcome in the economy without 1 or a (denoted $E_{(-1,-a)}$). In $E_{(-1,-a)}$, there is only one agent in the economy, thus agent 2 gets resource b for free and her utility is $\pi_{(-1,-a),2} = w_{2,b} p_{2,b} = 32$. The payment at which agent 2 is indifferent between getting a and getting $\pi_{(-1,-a),2}$ is therefore $q_{(-1,-a),2} = u_{2,a}^{-1}(\pi_{(-1,-a),2}) = 40$, thus agent 1 face a payment $z_{1,a} = \max(q_{(-1,-a),2}, 0) = 40$ on resource a .

Similarly, we can compute $z_{1,b} = 0$, $z_{2,a} = 30$ and $z_{2,b} = 0$. Under these payments, $u_{1,a}(z_{1,a}) = 8 < u_{1,b}(z_{1,b}) = 16$ thus agent 1 is allocated resource b . $u_{2,a}(z_{2,a}) = 34 > u_{2,b}(z_{2,b}) = 32$, thus agent 2 is allocated resource a . The total utilization under GCSP is therefore $p_{1,b} + p_{2,a} = 1.6$.

We can see in Example 8 that comparing with the VCG mechanism, which favors agents with high vertical intercept, agents with high zero-crossings and therefore shallower expected utility

functions are more likely to get allocated in GCSP. This can lead in turn to higher utilization. For agents with continuous types (e.g. the exponential type model), utilization is further boosted due to the penalty charged by the mechanism.

Still, VCG may provide higher utilizations in somewhat extreme examples where the number of agents is small and there is little competition.

Example 9 (Better Utilization under VCG). Consider an economy with the following two agents 1 and 2, two items a and b , and the $w_i p_i$ type model:

- Item a : $v_{1,a} = 200$, $p_{1,a} = 0.2$; $v_{2,a} = 37.5$, $p_{2,a} = 0.8$
- Item b : $v_{1,b} = 550$, $p_{1,b} = 0.1$; $v_{2,b} = 66.67$, $p_{2,b} = 0.6$

The expected utility curves of the two agents for the two resources are as shown in Figure 8b. Under VCG, agent 1 gets resource b and pays 10, agent 2 gets resource a and pays 0. Total utilization is $0.1 + 0.8 = 0.9$. Under GCSP, we can compute following the same steps as in Example 8 that agent 1 gets resource a and pays 0, agent 2 gets resource b and pays 16.67 for no show penalty. Total utilization is $0.2 + 0.6 = 0.8$, which is lower than that of VCG.

6 Simulation Results

In this section, we compare the utilization for different mechanisms under the exponential type model. We have found the simulation results to be very robust regarding type models and the distribution on types.⁶

6.1 Single Resource

Under the exponential type model (Example 7), the value for agent i to use the resource is fixed w_i and the opportunity cost is an exponentially distributed random variable with parameter λ_i . $\mathbb{E}[V_i] = w_i - \lambda_i^{-1}$ where λ_i^{-1} is the expected value of the opportunity cost.

For the simulations, we assume a simple distribution of types, where the opportunity cost is uniformly distributed $\lambda_i^{-1} \sim U[0, L]$, parameterized by $L > 0$ and given λ_i , the value is also uniformly distributed $w_i \sim U[0, \lambda^{-1}]$.

SP, CSP and Benchmarks We set $L = 10$, vary the number of agents from 2 to 15, compute the average utilization over 10,000 randomly generated profiles under SP, CSP and compare with the benchmarks.

See Figure 9. The *First best* benchmark is the highest achievable utilization subject to the IR and ND constraints as discussed in Example 7, assuming that the mechanism has full information about agents' types. *P1-P5 UB* is an upper bound on the utilization that can be achieved by any mechanism that satisfies (P1)-(P5). In particular, is the utilization when the agent with the highest zero-crossing point is allocated and charged her zero-crossing as the penalty, as we discussed in Section 4. *Random* is the average utilization achieved by randomly allocating the resource to one of the agents without charging any payment.

We can see from the figure that CSP achieves a significant improvement comparing with SP, and is performing relatively well compared with the P1-P5 UB. The utilization of CSP dominates that of SP on average as well as profile-by-profile (as proved in Theorem 2.)

⁶See Appendix D for derivation of bids in different models, and for a large sweep of simulation results.

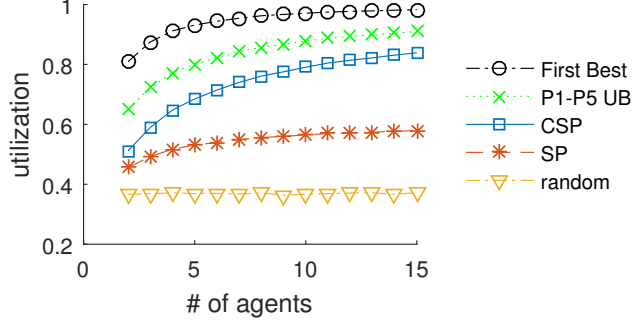


Figure 9: Utilization of a single resource under the exponential model: SP, CSP and benchmarks.

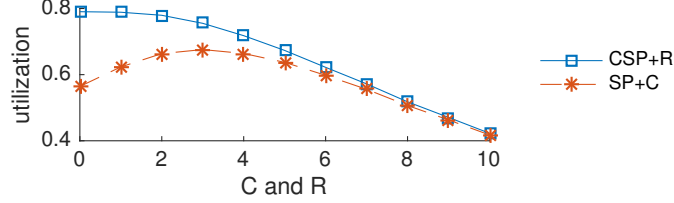


Figure 10: Utilization of a single resource under the exponential model: effect of reserve prices

Fixed Penalty and Reserve Prices Here, we fix the number of agents to be $n = 10$. Figure 10 examines utilization of CSP+R and SP+C mechanisms. We can see that while $C > 0$ may improve the utilization of SP, it is not enough to achieve higher utilization than CSP. Fixing $R = C$, the utilization of CSP+R not only achieves a higher average utilization than SP+C but dominates that of SP+C profile-by-profile, as we proved in Theorem 6.

CSP+R is best for zero reserve, and as R increases the average utilization decreases, which shows that the average effect of the improvement in utilization due to the additional penalty is smaller than the loss of utilization when the resource is not allocated because of R . Figure 11 also compares the utilization of CSP and CSP+R, profile-by-profile, for different reserve prices. For $R > 0$, then when the resource is allocated under CSP+R the utilization under CSP+R can be higher than under CSP. As R increases, there is a higher improvement in utilization if the resource is allocated under CSP+R, however, there are a lot more profiles in which the resource is not allocated under CSP+R. The overall effect is that the average utilization under CSP+R is low relative to $R = 0$.

6.2 Multiple Resources

We compare the performance of different mechanisms when the resources are not identical. For each agent i and alternative j , $V_{i,j}$ is of exponential type with parameters $w_{i,j}$ and $\lambda_{i,j}$ and we assume the opportunity costs $\lambda_{i,j}^{-1}$ are iid $\lambda_{i,j}^{-1} \sim U[0, L]$ and that given $\lambda_{i,j}$, the value $w_{i,j} \sim U[0, \lambda_{i,j}^{-1}]$.

We fix the number of resources to be $k = 3$ and the maximum opportunity cost $L = 10$. The average utilization over 10,000 randomly generated profiles is shown in Figure 12a as we vary the number of agents from 2 to 15. The *First best* and *Random* benchmarks are computed in the same way as in single resource assignment.

As the number of agents increases, the utilization of GCSP is boosted significantly since competition results in agents with higher probability of utilization begin selected and increases the penalties which further boost utilization. Unlike in the single resource case, the utilization under

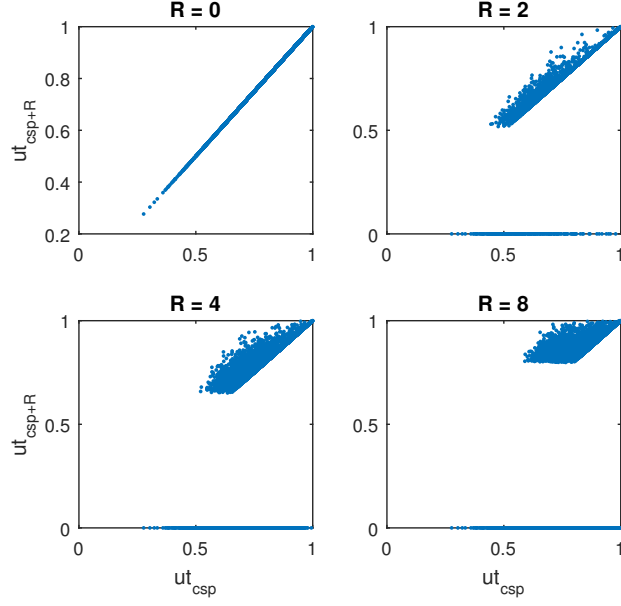


Figure 11: Utilization in CSP vs CSP+R with different reserve prices R for the exponential model.

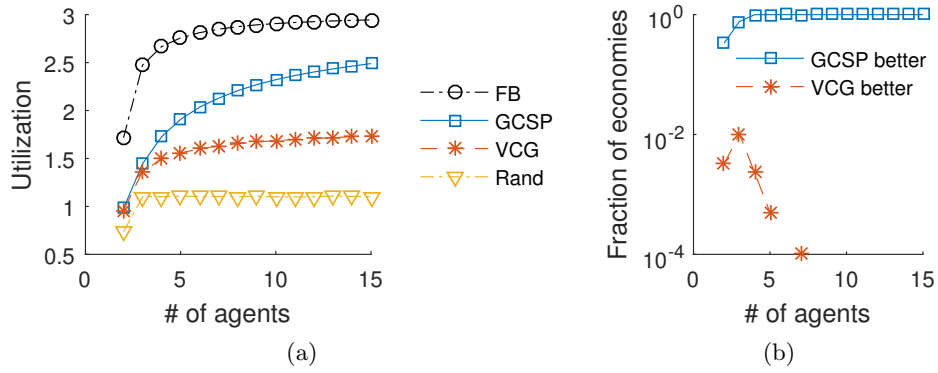


Figure 12: Utilization for allocating three resources under the exponential type model.

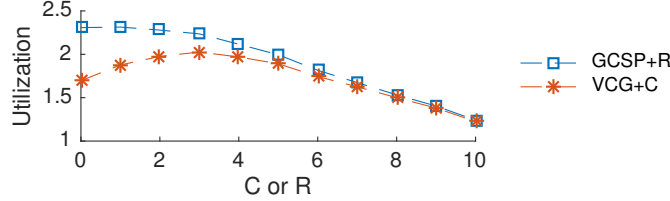


Figure 13: Allocating three resources under the exponential type model: Effect of reserve prices.

GCSP does not dominate that of VCG profile-by-profile. The fraction of profiles for which GCSP has strictly higher utilization, and for which VCG has strictly higher utilization are shown in Figure 12b. When the number of agents is very small, for about 1% of the profiles, VCG has strictly higher utilization than GCSP. When there are only 2 agents, GCSP has strictly better utilization than VCG about half the time, and GCSP is almost always better for $n \geq 5$.

The effect of reserve prices R for GCSP (the GCSP+R) and fixed penalty C for VCG (the VCG+C) is shown in Figure 13, where the number of agents n is fixed to be 10 and the reserve R and penalty C vary from 0 to 10. There is a very small improvement in utilization for the GCSP+R mechanism comparing with no reserve when R is close to 0, otherwise the patterns are similar as in the case of single resource assignment.

7 Conclusion

We have introduced the coordinated assignment problem in which the goal is to maximize the expected utilization of resources. We prove that the contingent-second price mechanism maximizes utilization for single unit and multiple, identical unit allocation problems across a wide class of mechanisms. The idea is that bids convey information about the reliability of an agent. We also showed via simulation that a generalized, contingent-payment mechanism has good performance for the utilization-maximization version of the classical unit-demand assignment problem.

Interesting directions for future work include weakening the assumptions on the type space for the optimality of CSP, and developing theoretical results for the multiple, heterogeneous resources. Another direction is expanding the analysis to richer domains where coordination success or failure depends on interactions between actions rather than a count of how many agents follow through and take the intended action. Finally, since contingent mechanisms act as *soft commitment devices* [4], we conjecture that they are effective for agents with present-bias and other behavioral models.

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Appendix

We provide missing proofs, discussions and examples in Appendix A. We discuss the welfare interpretation of our problem and the optimality of the SP+C mechanism without the no-deficit or no-charge requirements in Appendix B.

The two-dimensional payment space is discussed in more detail in Appendix C, followed by the definition and analysis of a class of DSIC agent-dependent mechanism termed contingent menu mechanisms (CMM). Appendix D provides detailed derivations for the expected utilities, DSE bids and distribution of bids for different type models and type distributions, additional simulation results on different type models, type distributions and the CMM's.

A Proofs

Lemma 1 (Properties of expected utility). Under (A2), the expected utility $u(z)$ as a function of the penalty z satisfies:

1. $u(z) = \mathbb{E}[V^+] - \int_0^z F(-v)dv$,
2. $u(0) = \mathbb{E}[V^+]$, and $\lim_{z \rightarrow \infty} u(z) = \mathbb{E}[V]$, and
3. $u(z)$ is continuous, convex and monotonically decreasing w.r.t. z .

Proof.

1. From integration by parts:

$$\begin{aligned}
 u(z) &= \mathbb{E}[V^+] + \mathbb{E}[V \cdot \mathbf{1}\{-z \leq V \leq 0\}] - z\mathbb{P}[V < -z] \\
 &= \mathbb{E}[V^+] + \int_{-z}^0 v dF(v) - zF(-z) \\
 &= \mathbb{E}[V^+] + vF(v)|_{-z}^0 - \int_{-z}^0 F(v)dv - zF(-z) \\
 &= \mathbb{E}[V^+] - \int_0^z F(-v)dv
 \end{aligned}$$

2. It's easy to know $u(0) = \mathbb{E}[V^+]$ from (4). For the 2nd part, consider the following two cases:

- 1) $\mathbb{E}[V] = -\infty$. We know that $\int_{-z}^0 v f(v)dv \rightarrow -\infty$ as $z \rightarrow \infty$. Thus $u(z) \rightarrow -\infty$ since $\mathbb{E}[V^+] < \infty$ and $z\mathbb{P}[V < -z] < 0$.
- 2) $\mathbb{E}[V] > -\infty$. In this case V is absolutely integrable. Define $V^- = \min\{V, 0\}$, we from the linearity of expectation $\mathbb{E}[V] = \mathbb{E}[V^+] + \mathbb{E}[V^-]$ and from the definition of infinite integration $\mathbb{E}[V^-] = \lim_{z \rightarrow \infty} \left[\int_{-z}^0 v f(v)dv - z\mathbb{P}[V < -z] \right]$. Thus the claim holds.

3. We know from (4) that the first part of $u(z)$ is a constant and the 2nd part is an integration of a bounded function ($|F(v)| \leq 1$). Thus $u(z)$ is *continuous*.

Taking the derivative of $u(z)$ w.r.t. z we can get

$$\frac{d}{dz}u(z) = -\frac{d}{dz} \int_0^z F(-v)dv = -F(-z) < 0$$

Thus $u(z)$ is *monotonically decreasing*. It's easy to see that $\frac{d}{dz}u(z)$ is monotonically increasing as z increases, thus $u(z)$ is (weakly) convex. □

Theorem 6 (CSP+R Dominates SP+C). *Let $\{F_i\}_{i \in N}$ be any set of bidders' value distributions with $\mathbb{E}[V_i] < 0$. In the unique dominant strategy equilibrium, the utilization of SP+C is dominated by that of CSP+R while R is set to be equal to C .*

Proof. Since we set the secret reserve price of CSP as C , let's denote this mechanism as CSP+C. Recall that CSP+C allocates the resource to the same agent that gets allocated under CSP, as long as the reserve price is met. We need to consider the following cases:

1. SP+C selects allocated the resource to a different agent from that of CSP. Assume that agent 1 was allocated under SP+C and agent 2 was allocated under CSP+C, we must have $u_1(C) \geq u_2(C)$ and $z_1^0 \leq z_2^0$. With the same argument as the proof of Thm. 4, we can show that SP+C has worse utilization than CSP+C for this case.
2. SP+C selects the different allocated agent as CSP. If C is lower than the second highest bid, the SP+C allocated agent faces an extra payment C contingent on no-show whereas the CSP+C allocated agent faces a higher contingent payment. Since the utilization increases with the contingent payment, CSP+C has higher utilization.
3. If C is higher than the second highest bid, the contingent payment facing the allocated agent would be the same under both mechanisms, thus they have the same utilization.
4. If C is higher than the highest bid from the agents, the resource would not be allocated in either mechanism, thus the utilization would both be zero.

□

Theorem 7 (Dominant Strategy under γ -CSP). *Under the γ -CSP mechanisms, it is a dominant strategy for an agent to bid the summation of the coordinates of the crossing point between her zero-profit curve and $y = \gamma/(1 - \gamma)z$.*

Proof. Fix agent i , and assume that the highest bids from the rest of the agents is $b_{i'}$. First of all, note that the γ -CSP mechanism is a critical payment mechanism: agent i is allocated if and only if $b_i \geq b_{i'}$. Denote the crossing point between agent i 's zero-profit curve and $y = \gamma/(1 - \gamma)z$ as point A . We only need to prove that a DRM version of γ -CSP which allocates to agent i iff $z_A + y_A \geq b_{i'}$.

Observe that if agent i gets allocated, the two part payment that she face would be:

$$\begin{aligned} z_i &= t_i^{(0)} - t_i^{(1)} = b_{i'} - \gamma b_{i'} = (1 - \gamma)b_{i'} \\ y_i &= t_i^{(1)} = \gamma b_{i'} \end{aligned}$$

Since $y_A = \frac{\gamma}{1-\gamma}z_A$ always holds, (z_i, y_i) resides below the zero-profit curve $u_i(z, y) = 0$ iff $z_A + y_A \geq z_i + y_i = b_{i'}$. Thus the DRM is agent-maximizing. □

Lemma 3 (Possible Allocation and Payments). *For any mechanism that satisfies (P1), (P2), (P4) and (P5), the zero-profit curve of the allocated agent i^* must be a part of the frontier of all agents, and the two-part payment (z^*, y^*) facing the allocated agent must be weakly above the frontier of the rest of the economy.*

Proof. We only need to show that in any mechanism satisfying (P1)-(P5), the existence of an unallocated agent $i' \neq i^*$ s.t. $u_{i'}(z^*, y^*) > 0$ leads to a contradiction. Consider the economy E depicted in Figure 14. Assume that agent 1 is allocated the resource and charged two-part payment at point A : $(z_1, y_1) = (z_A, y_A)$, s.t. $u_2(z_1, y_1) > 0$.

Now consider another economy E' , where agent 1 is replaced by agent $1'$, whose type is equivalent to that of agent 2: $u_{1'}(z, y) = u_2(z, y)$ for all (z, y) . We have showed in the main text that in economy E' , with deterministic tie-breaking, at least one of agents 2 and $1'$ is not getting allocated. It would be a useful deviation for the unallocated agent to report u_1 as her type, get allocated and get positive utility. Now consider random tie breaking, in which case agents 2 and $1'$ each getting allocated with probability 0.5.

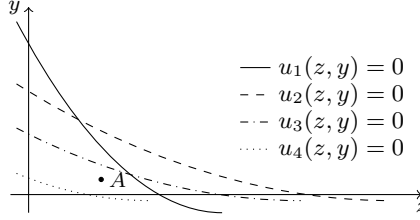


Figure 14: Illustration for the proof of Lemma 3.

In economy E' , both agent $1'$ and agent 2 must get utility at least $u_2(z_A, y_A)$, otherwise, the one that does not get at least $u_2(z_A, y_A)$ would have incentive to strategize and report u_1 as her type, get charged (z_A, y_A) , thus gets better off. This violates DSIC. Since we require the mechanism to be anonymous, it can only break ties uniformly at random between agent $1'$ and agent 2 since they have identical types, therefore, both agents would get allocate the resource with probability 1/2, and get charged some other payment, say z_B, y_B s.t. $u_2(z_B, y_B) \geq 2u_2(z_A, y_A)$. This implies that the point (z_B, y_B) must reside weakly below $u_2(z) - 2u_2(z_A, y_A)$.

Before we proceed, recall from Lemma 4 that in order to satisfy the no-deficit condition, the payment that the allocated agent face must have a non-negative y component. We further observe that the payment (z, y) facing the allocated agent must reside above the $y = -z$ line, which is above the $y = 0$ line when z is negative. Otherwise, assume that there exists an allocated agent in an economy facing payment (z, y) where $z < 0$ and $y < -z$. It's easy to construct an agent with (w_i, p_i) type that has to be allocated and for whom the expected revenue is negative, as we did for the proof of Lemma 4.

Now we consider the following cases. First, if $u_2(0) - 2u_2(z_A, y_A) < 0$. It's easy to check that in this case, the payment (z_B, y_B) that resides below $y = \max(-z, 0)$ thus the mechanism does not satisfy ND. This is a contradiction.

Now we assume that $u_2(0) - 2u_2(z_A, y_A) \geq 0$. In this case it is possible for (z_B, y_B) to reside weakly below $u_2(0) - 2u_2(z_A, y_A)$ and weakly above $y = \max(-z, 0)$ at the same time. Let us now construct another agent, say agent $1''$, such that $u_{1''}(z) = \max(-z, u_2(z) - \frac{3}{2}u_2(z_A, y_A))$ for all z (it's easy to check that there must be one such type among all CDF's that satisfy (A1) - (A3)). It must be the case that $u_{1''}(z_B, y_B) \geq \frac{1}{2}u_2(z_A, y_A)$, since at z_B , $u_{1''}(z)$ must be above $-z$ thus equal to $u_2(z) - \frac{3}{2}u_2(z_A, y_A)$, otherwise ND is violated.

Consider the economy E'' which is identical to E except that agent 1 is replaced by agent $1''$. We know that agent $1''$ must be allocated and get expected utility at least $0.5 * \frac{1}{2}u_2(z_A, y_A) = \frac{1}{4}u_2(z_A, y_A)$, otherwise, she will have the incentive to report the type of agent 2 as her type, get allocated with probability 0.5, get charged (z_B, y_B) and gain $\frac{1}{4}u_2(z_A, y_A)$.

Since the mechanism is deterministic, we know that in this case, agent 2 must be allocated with

probability zero since agent 2's type is different from that of agent 1" thus the mechanism cannot randomize between the two. We know that in this case $u_2(z) \geq u_{1''}(z)$ for all z , thus the resource is allocated to an agent that is dominated by the other. With a similar argument, we can show that in an economy where agent 1" is replaced by another agent with the type of agent 2, both agents with the type of agent 2 must get expected utility at least $\frac{1}{4}u_2(z_A, y_A) + \frac{3}{2}u_2(z_A, y_A) = \frac{7}{4}u_2(z_A, y_A)$, what they can get by reporting the type of agent 1 as their own. As they both get allocated with probability 0.5, the payment that they face, call it (z_C, y_C) , must reside below $u_2(z) - \frac{7}{2}u_2(z_A, y_A)$.

Finally, let's consider an economy with agent 1''' and agent 2 and the rest of the agents, where $u_{1'''}(z) = \max(-z, u_2(z) - 3u_2(z_A, y_A))$. It's easy to see that the utility curve of agent 1''' is dominated by that of agent 1''. If agent 1''' is allocated, agent 1'' will have a useful deviation since any IR payment for agent 1''' would give agent 1'' a higher expected utility than $\frac{1}{4}u_2(z_A, y_A)$. However, if agent 1''' is not allocated, she has an incentive to report the type of agent 2 and benefit, therefore violates DSIC.

Thus we have showed that all agents that could be allocated must be at the frontier of all agents at the point above the payment pair point. □

Theorem 4. *For any set of agent types satisfying (A1)-(A3), CSP achieves highest utilization among all mechanisms that satisfy (P1)-(P6) and where the payments are agent-independent.*

Proof. Payments under the γ -CSP mechanism resides on the $y = \frac{\gamma}{1-\gamma}z$ line, and it is a dominant strategy for an agent to bid the summation of the two coordinates of the point at which $y = \frac{\gamma}{1-\gamma}z$ and her zero-profit curve crosses (which always exists and is unique). It's easy to see that γ -CSP allocates the resource to the agent at the crossing point of $y = \frac{\gamma}{1-\gamma}z$ and the frontier (e.g. agent 2 in the economy depicted in Figure 7) and charges the allocated agent the crossing point of $y = \frac{\gamma}{1-\gamma}z$ and the second frontier $((z_2, y_2) = (z_D, y_D))$ in the Figure 7 example).

As γ decreases from 1 to 0, the penalty part z of the crossing point of $y = \frac{\gamma}{1-\gamma}z$ and the second frontier increases, thus so is the utilization $\mathbb{P}[V_{i^*} \geq z] = \frac{d}{dz}\bar{u}_N(z) + 1$ due to the convexity of $\bar{u}_N(z)$. Thus CSP which corresponds to 0-CSP achieves the highest utilization for any economy and in particular dominates SP. □

Theorem 8 (CSP is not dominated under (P1)-(P5)). *Assume the type space is the set of all value distributions satisfying (A1)-(A3). The CSP mechanism is not dominated by any mechanism that satisfies (P1)-(P5).*

Proof. Assume that there is a mechanism M s.t. the utilization under M is always as good as that of CSP for any economy. We will show that M must be identical to CSP.

- (1) We know from Lemma 3 that in any mechanism that satisfy (P1)-(P5), the allocated agent must be on the frontier, thus any mechanism that dominate CSP must also allocate to the agent that gets allocated in CSP, ignoring tie-breaking issues for now. Note that this rules out the use of reserve prices, which might leave the resource unallocated.
- (2) Consider an economy E and the allocated agent in E under mechanism M , say agent 1. In order for mechanism M to dominate CSP in utilization, agent 1 must face a penalty at least as high as the second highest zero-crossing (let's denote it as z_2^0). If the penalty is indeed z_2^0 , then the outcome coincides with that of CSP, and we are all set.

- (3) Now assume that agent 1 is charged a payment (z_1, y_1) where the penalty z_1 is higher than the second highest zero-crossing z_2^0 . We know from Lemma 4 that the payment must be above the frontier of the rest of the agents (the “second frontier”). In order to stay weakly below the second frontier and have a larger penalty, the base payment must be negative, which would result in violation of BB as shown in the lemma.
- (4) Consider now agent 1', whose expected utility dominates the frontier of agents 2- n , but satisfy $u_{1'}(z_1, y_1) < 0$. We know from Lemma 3 that she is the only agent that can be allocated in economy E' with agents 1', 2, ..., n , and she has to be charged another two-part payment $(z_{1'}, y_{1'}) \neq (z_1, y_1)$ since the later is not IR. We can prove that if $(z_{1'}, y_{1'}) = (z_2^0, 0)$ which is the second highest zero-crossing, the mechanism is not DSIC for agent 1 in the original economy since $(z_{1'}, y_{1'})$ is a better payment than (z_1, y_1) . If $(z_{1'}, y_{1'})$ is not the second highest zero-crossing, it must reside above the horizontal axis, weakly below the second highest frontier, thus we can construct another agent 1'' with $w_i p_i$ type that gets positive utility at $(z_{1'}, y_{1'})$ but has a smaller zero-crossing than $(z_2^0, 0)$. In this economy, agent 1'' must be allocated, and the utilization is worse than that of CSP.

□

A.1 Effect of Reserve Prices

We now show that for the simple exponential model and uniform model (see Appendix D), with simple type distributions as follow, reserve prices do not help with the utilization of a single resource under the CSP+R mechanism. Reserve prices for a single agent auction is equivalent to a posted penalty.

Proposition 1 (Reserve Price Do not Help for One Agent). *For the economy with $n = 1$ agent with the two type models with corresponding type distributions as follow:*

1. *Exponential model, with $V = w - O$ where O is the opportunity cost, an exponential variable with parameter λ . Let λ^{-1} be uniformly distributed on some interval $\lambda^{-1} \sim U[0, \Lambda^{-1}]$ for $\Lambda^{-1} > 0$ and $w \sim U[0, \lambda^{-1}]$.*
2. *Uniform model, with $V \sim U[-a_1, a_2]$, where we have parametrized distribution $a \sim U[0, 1]$ and $b \sim U[0, a_1/2]$, and let the $a_2 = b$ and let $a_1 + a_2 = a \Rightarrow a_1 = a - b$.*

The optimal reserve price for achieving highest utilization under the CSP+R mechanism is $R = 0$, i.e. reserve prices do not help.

Proof. To prove is that, for each type model and type distribution, the expected utilization gain by imposing a reserve penalty r when the agent still accepts the resource is smaller than the expected utilization loss when r is larger than the agent's zero-crossing point and the resource is not allocated. We prove this for both models.

1. Exponential model. Denote $\eta = \lambda^{-1}$. For the combination of w , η and b s.t. the item is not allocated, the loss of utilization comparing with no reserve price is

$$1 - e^{-\lambda(w+0)}.$$

When the resource is allocated even when the reserve price is set to b , the improvement in utilization is:

$$1 - e^{-\lambda(w+b)} - (1 - e^{-\lambda(w+0)}) = e^{-\lambda w} - e^{-\lambda(w+b)}$$

Therefore, on average, the expected gain of utilization when the reserve price is set to b would be

$$\begin{aligned}
& G(b) \\
&= \int_0^{\Lambda^{-1}} \left[\int_w^\eta \left(-\mathbb{1}\{b_{csp}(\eta, w) \leq b\}(1 - e^{-\lambda w}) + \mathbb{1}\{b_{csp}(\eta, w) \geq b\}(e^{-\lambda w} - e^{-\lambda(w+b)}) \right) \frac{1}{\eta} dw \right] \frac{1}{\Lambda^{-1}} d\eta \\
&= \Lambda \int_0^{\Lambda^{-1}} \frac{1}{\eta} \left[- \int_0^{\eta + \eta W(-e^{-1-b/\eta})} (1 - e^{-\lambda w}) dw + \int_{\eta + \eta W(-e^{-1-b/\eta})}^\eta (e^{-\lambda w} - e^{-\lambda(w+b)}) dw \right] d\eta \\
&= \Lambda \int_0^{\Lambda^{-1}} (-1/e + e^{-1-b/\eta}) d\eta = \frac{1}{e} \left[-1 + e^{-b\Lambda^{-1}} - b\Lambda^{-1}\Gamma(0, b\Lambda^{-1}) \right]
\end{aligned}$$

It's easy to see that $G(0) = 0$, which corresponds to the case of having no reserve price. We can also see that the derivative of $G(b)$ w.r.t b

$$G'(b) = -\frac{1}{e}\Lambda^{-1}\Gamma(0, b\Lambda^{-1}) < 0$$

since $\Gamma(0, b\Lambda^{-1}) > 0$ for all b . This shows that $G(b) < 0$ for all $b > 0$, thus reserve prices would not be helpful for the exponential type either. The analytic form and the simulation results for the effect of reserve prices are as shown in Figure 15

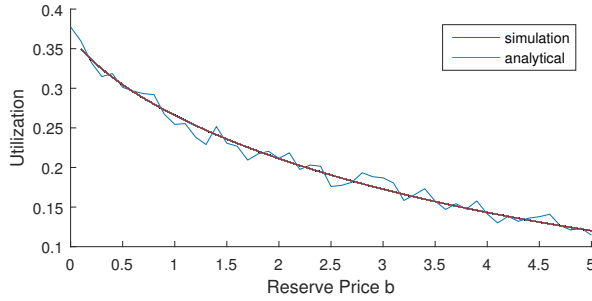


Figure 15: The effect of reserve prices for exponential type.

2. Uniform model. We know that the zero-crossing, which is the CSP bid for this an agent with uniform model as described above would be:

$$b_{csp}(a, b) = a - b - \sqrt{(a - b)^2 - b^2} = a - b - \sqrt{a^2 - 2ab} \leq a/2$$

therefore for $a < 2r$, we lose all utilization, which is $1/4$ in expectation. For $a \geq 2r$ and for a certain b , the loss of utilization if the resource is allocated would be $\frac{b}{a}$. and the gain from reserve price r if the resource is still allocated would be $\frac{r}{a}$. The gain in utilization while reserve is R is therefore:

$$\begin{aligned}
G(r) &= -\frac{2r}{1} \frac{1}{4} + \int_{2r}^1 \int_0^{a/2} \left[-\mathbb{1}\{b_{csp}(a, a) < b\} \frac{b}{a} + \mathbb{1}\{b_{csp}(a, b) \geq b\} \frac{r}{a} \right] \frac{2}{a} db da \\
&= -\frac{r}{2} + \int_0^1 \frac{2}{a} \left[- \int_0^{\sqrt{2ar-r}} \frac{b}{a} db + \int_{\sqrt{2ar-r}}^{a/2} \frac{r}{a} db \right] da \\
&= r(-r + \log(2) + \log(r))
\end{aligned}$$

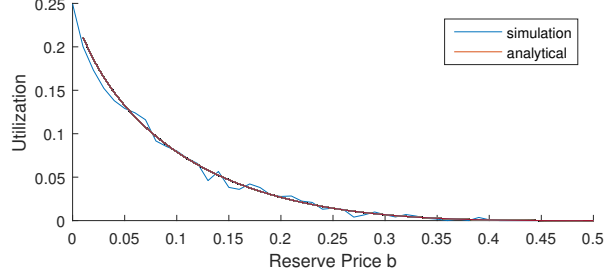


Figure 16: The effect of reserve prices for uniform type.

To show analytically that the gain is always negative, note that

$$G'(r) = 1 - 2r + \log(2) + \log(r)$$

and

$$G'(r) = -2 + \frac{1}{r} \geq 0$$

for all $0 < r \leq 1/2$. It's also easy to check that $G(0) = 0$ and $G(1/2) = -0.25$, thus $G(r) \leq 0$ always hold which means that the reserve price is never helpful.

□

B Welfare Considerations

Denote $aw(r) \triangleq \mathbb{E} \left[V_{i^*} \cdot \mathbb{1}\{V_{i^*} \geq t_{i^*}^{(1)}(r) - t_{i^*}^{(0)}(r)\} \right]$ as the welfare that the agent gets from using the resource. Consider an alternative problem, where instead of maximizing the utilization, there is instead a fixed value C for the society if the resource is utilized, and that the mechanism designer tries to maximize the social welfare:

$$sw(r) = aw(r) + C \cdot ut(r).$$

Fix an agent i . From the perspective of maximizing social welfare, the agent should use the resource iff $v_i \geq -C$, which brings total value $C + v_i \geq 0$ to the society. Thus the highest possible welfare if the resource is allocated to agent i is

$$sw_i = \mathbb{E} [V_i \cdot \mathbb{1}\{V_i \geq -C\}] + C \cdot \mathbb{P}[V_i \geq -C] = b_{i,SP+C}^* + C$$

This optimal decision can be induced by charging a no-show penalty of C , thus if C is small and $b_{i,SP+C}^* \geq 0$ for all i , the SP+C mechanism maximizes social welfare under the alternative model since the allocated agent $i^* \in \arg \max_{i \in N} b_{i,SP+C}^* = \arg \max_{i \in N} sw_i$.

When C is large, e.g. the value of the resource being utilized for the society dominates the value of the resource to the agents, $b_{i,SP+C}^*$ would be negative and agents stop participating in SP+C. Generalizing the SP+C mechanism to allow negative bids and negative payments (i.e. allowing payments from the mechanism to the agents), we can show that the mechanism is still welfare-maximizing.

Theorem 9 (Optimality of SP+C). *Let $\{F_i\}_{i \in N}$ be any set of bidders' value distributions. Under the alternative model where no-show causes a cost C for the society, the SP+C mechanism is DSIC, IR, and optimal for welfare, if we allow agents to submit negative bids.*

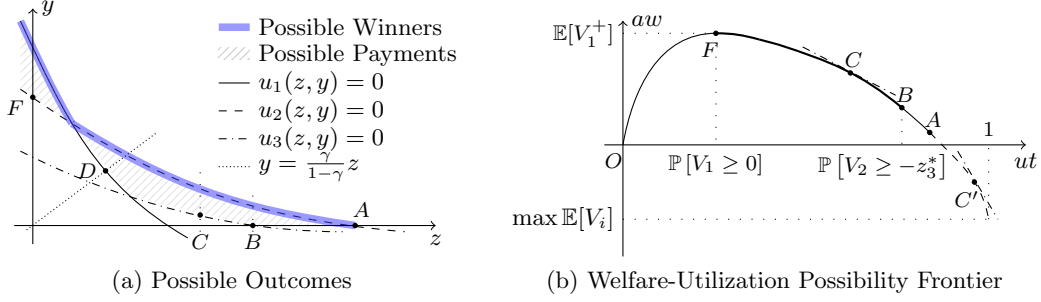


Figure 17: Characterization of possible outcomes and utilization for mechanisms satisfying (P1)-(P5).

Proof. We proceed in steps.

1. Recall from the analysis in Section 3, agents would bid

$$b_{i,SP+C}^* = \mathbb{E}[V_i \cdot \mathbb{1}\{V_i \geq -C\}] - C \cdot \mathbb{P}[V < -C]$$

which is exactly the highest welfare if the resource is allocated to one of the agents. Since the highest bidder is allocated the resource, the allocation is welfare maximizing, and the penalty C guarantees that the resource is used by the agent to maximize welfare.

2. Note that for large social cost C , there might not be an agent with non-negative $b_{i,SP+C}^*$ thus no rational agent would participate in this mechanism, resulting in zero utilization, which is weakly worse than randomly allocating the resource to an agent.
3. However, in this case the resource is more of a “burden” and it’s not unnatural to allow the mechanism to pay agents for participating. It is easy to show that when negative bids are allowed, it is still a dominant strategy for agents to bid $b_{i,SP+C}^*$ which is lower bounded by $-C$, so that the welfare is maximized with the mechanism paying the allocated agent at most C .

□

However, as we have discussed in Section 4, the mechanism would not be no deficit. We show via the following analysis that when this is the case welfare is maximized by maximizing utilization, which again is the objective of our original model.

The set of all possible outcomes under assumptions (P1)-(P5) has been characterized in Section 4. As the penalty part of the payment facing the allocated agent moves from $-\infty$ to z_N^0 , the welfare-utilization possibility frontier under (P1)-(P5) is as depicted in Figure 17b, with outcomes under corresponding mechanisms labeled by the same letter as in the discussions in Section 3. The solid segment is achievable under (P1)-(P5), and if no-deficit is relaxed we are able to go beyond A and achieve the dashed curve.

When the penalty $z = -\infty$, i.e. paying agents an infinite amount for no-show, the agent never uses the resource thus $aw(-\infty) = ut(-\infty) = 0$. As z increases from $-\infty$ to 0 , $aw(z)$ and $ut(z)$ both increase, and at $z = 0$ agent welfare is maximized (point F , achieved by SP). As z increases from 0 to ∞ , $ut(z)$ increase and $aw(z)$ decreases, until when $z = \infty$, the resource is allocated to the agent with highest $\mathbb{E}[V_i]$ (the agent on the frontier at $z \rightarrow \infty$ and utilization approaches 1). At each point of the curve, the slope of the $aw \sim ut$ curve equals to the penalty z that facing the allocated agent. The Solid line is all possible outcomes under (P1)-(P5), and if we relax the ND or IR constraint, we are able to achieve the dashed line, boosting utilization to 1.

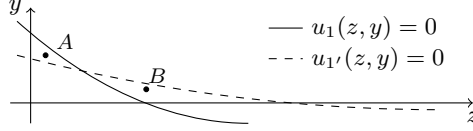


Figure 18: Two possible, zero iso-profit curves for an agent, where z is the penalty and y the base payment. For type F_1 and iso-profit curve $u_1(z, y) = 0$, agent 1 is allocated and charged two-part payment A . For type F'_1 and iso-profit curve $u_{1'}(z, y) = 0$, agent 1 is allocated and charged two-part payment B . Because $u_1(z_A, y_A) \geq u_1(z_B, y_B)$ and $u_{1'}(z_A, y_A) \leq u_{1'}(z_B, y_B)$, no agent has incentive to pretend to be the other, thus this remains truthful.

Points C , B in Figure 17b correspond to the outcome under SP+C, and CSP respectively, and A is the upper bound under (P1)-(P5). The outcome under SP+C is the point at which $aw \sim ut$ is tangent to the line $aw + C \cdot ut = \text{const}$, which maximizes the social welfare $sw = aw + c \cdot ut$ if the society derives utility C whenever the resource is utilized. When C is large, the tangent point may fall on the dashed segment of the curve (e.g. point C') thus the social welfare maximizing outcome cannot be achievable by any mechanism under (P1)-(P5).

This shows that our problem cannot be trivially solved, by designing mechanisms to maximize $sw = aw + C \cdot ut$, and then send $C \rightarrow \infty$.

C Agent-Dependent Mechanisms

In traditional auction problems where payments are one-dimensional, agent-independence is a necessary condition for DSIC, assuming agents' utilities decrease as the payments increase. Assume otherwise s.t. there exists an economy where agent 1 gets allocated and charged t . Replacing agent 1 with agent $1'$, $1'$ also gets allocated and is charged t' . Without loss of generality, assume $t < t'$, agent $1'$ would have incentive to report as agent 1, get allocated the resource, pay t and get $u_{1'}(t) > u_{1'}(t')$. What is critical here, is that there is an order on the set of all possible payments an agent could be charged, s.t. all agents agree on. However, agent-independence is not a necessary condition for DSIC because agents have heterogeneous preferences when payments reside in a two-dimensional space, as shown by the example in Figure 18.

Before jumping in to talk about possible agent dependent mechanisms that we can construct, we should note if the two-part payments are constrained on a domain on which s.t. there exists an order that all agents agree on, agent-independence becomes a necessary condition for DSIC to hold, as in the one dimensional payment space. All mechanisms we have talked about so far restricts payments in some way s.t. agent-independence must hold:

- SP: payments are restricted to $\{(z, y) | z = 0\}$, and the utility of any type monotonically decreases as y increases.
- SP+C: payments are restricted to $\{(z, y) | z = C\}$, and the utility of any type monotonically decreases as y increases.
- CSP: payments are restricted to $\{(z, y) | y = 0\}$, and the utility of any type monotonically decreases as z increases.
- γ -CSP: payments are restricted to $y = \frac{\gamma}{1-\gamma}z$, and the utility of any type monotonically decreases as z (and also y , since y is a monotonically increasing function of z), increases.

C.1 Contingent Menu Mechanisms

The example in Figure 18 shows the possibility to charge allocated agents different payments that depend on their reported types, as long as the payment is “agent-maximizing” for each allocated agent s.t. there is no incentive to report as another type. Inspired by the critical payment mechanisms for cases when payments are one dimensional, we now introduce a class of mechanisms which we call *contingent menu mechanisms* (CMM), which offers a menu of payments to choose from, or, as a DRM, chooses from a menu of payments for each allocated agent the payment that maximizes the utility for the agent, bases on her own report. The menu of payments that we offer, however, would not depend on the agent’s own report.

We denote the set of payment menus associated with each CMM as Π where each menu $\pi \in \Pi$ is a subset of $\mathbb{R}^2$ and is not necessarily singleton. First we state a few properties for the price menu set:

- B1. Each menu $\pi \in \Pi$ is defined as a *closed* subset of $\mathbb{R}^2$ s.t. we can define the utility of an agent with utility function $u(z, y)$ at price menu π as:

$$u(\pi) = \sup_{(z, y) \in \pi} u(z, y)$$

Intuitively, we choose the “agent-maximizing” price out of each menu, according to the reported utility function of the agent.

- B2. (Unique zero-crossing) For an agent with any type and the corresponding utility function $u(z, y)$, there exists a unique menu $\pi \in \Pi$ s.t.

$$u(\pi^*) = \sup_{(z, y) \in \pi^*} u(z, y) = 0 \quad (7)$$

- B3. (Monotonicity) There is a total order, say $\succeq$ defined on the set Π . For any $\pi_1, \pi_2 \in \Pi$ satisfying $\pi_1 \succeq \pi_2$, we must have

$$u(\pi_1) \leq u(\pi_2)$$

for any utility function u induced by any CDF satisfying B1 - B3.

Since we focus on dominant strategy equilibrium, we consider direct-revelation mechanisms w.l.o.g. Denote the set of agents as $N = \{1, 2, \dots, n\}$ and a reported type profile as $\hat{u} = (\hat{u}_1, \dots, \hat{u}_n)$ (recall that utility curves uniquely determine agents’ types), we now formally define the mechanism:

Definition 8 (Contingent Menu Mechanism). A contingent menu mechanism is equipped with a set of menus Π satisfying B1-B3 introduced above, with the following rules:

- *Allocation rule*: allocate the resource to the agent with the “highest” menu according to $\succ$:

$$x(\hat{u}) = i^* = \arg \max_{i \in N} \pi_i^*$$

where π_i^* is the unique zero-crossing menu of agent i .

- *Payment rule*: charge the allocated agent the payment in the second highest menu s.t. her utility is maximized:

$$t_{i^*}(\hat{u}) = \arg \max_{(z, y) \in \max_{i \neq i^*} \pi_i} \hat{u}_{i^*}(z, y)$$

When menus are singleton sets, the corresponding CMM is actually agent-independent, therefore we are able to interpret some of the mechanisms that we had been talking about as CMM's. See the following example:

Example 10 (Agent-independent CMM's).

- Second Price Auction (SP)
 - The set of payments are singleton sets parameterized by non-negative real numbers: $\Pi = \{\pi_\alpha\}_{\alpha \in \mathbb{R}_{\geq 0}}$ where $\pi_\alpha = \{(0, \alpha)\}$.
 - The total order $\succeq$ is defined as: $\pi_{\alpha_1} \succeq \pi_{\alpha_2}$ iff $\alpha_1 \geq \alpha_2$.
- Second Price with fixed penalty C (SP+C):
 - The set of payments are singleton sets parameterized by real numbers: $\Pi = \{\pi_\alpha\}_{\alpha \in \mathbb{R}}$ where $\pi_\alpha = \{(C, \alpha)\}$.
 - The total order $\succeq$ is defined as: $\pi_{\alpha_1} \succeq \pi_{\alpha_2}$ iff $\alpha_1 \geq \alpha_2$.
- Contingent Second Price (CSP)
 - The set of payments are singleton sets parameterized by non-negative real numbers: $\Pi = \{\pi_\alpha\}_{\alpha \in \mathbb{R}_{\geq 0}}$ where $\pi_\alpha = \{(\alpha, 0)\}$.
 - The total order $\succeq$ is defined as: $\pi_{\alpha_1} \succeq \pi_{\alpha_2}$ iff $\alpha_1 \geq \alpha_2$.
- γ -CSP (as defined in Appendix B)
 - The set of payments are singleton sets parameterized by non-negative real numbers: $\Pi = \{\pi_\alpha\}_{\alpha \in \mathbb{R}_{\geq 0}}$ where $\pi_\alpha = \{((1 - \gamma)\alpha, \gamma\alpha)\}$.
 - The total order $\succeq$ is defined as: $\pi_{\alpha_1} \succeq \pi_{\alpha_2}$ iff $\alpha_1 \geq \alpha_2$.
- Arbitrary payment independence mechanism:
 - The set of payments are singleton sets parameterized by non-negative real numbers: $\Pi = \{\pi_\alpha\}_{\alpha \in \mathbb{R}_{\geq 0}}$ where $\pi_\alpha = \{(z(\alpha), y(\alpha))\}$ and z and y are continuously monotonically increasing functions of α .
 - The total order $\succeq$ is defined as: $\pi_{\alpha_1} \succeq \pi_{\alpha_2}$ iff $\alpha_1 \geq \alpha_2$.

Note that 1) in SP+C, we are allowing agents' zero-crossing menu to be parameterized by negative numbers, i.e. submitting negative bids, which enables the mechanism to be welfare maximizing as discussed in Appendix B and 2) in order to satisfy B1-B3, $z(\alpha)$ and $y(\alpha)$ do not necessarily need to be increasing with α . We now prove that all contingent menu mechanisms are truthful and do satisfy many desired properties.

Theorem 10. *Any contingent menu mechanism defined above satisfies (P1), (P2), (P4)-(P6).*

Proof.

- Truthfulness and IR follows from the same argument for that of the second price auctions: utilities are monotonically decreasing as menus increase thus it is a dominant strategy to report the zero-crossing, which is guaranteed to be unique.

- It is easy to see that the mechanism is deterministic and anonymous, and never left the resource unallocated.

□

We should note, however, we are not restricting the domain for the payment menus thus the mechanism is not necessarily BB. This could be easily fixed by restricting the prices to the first quadrant and not including mechanisms like SP+C.

It might not be clear whether a CMM with non-singleton menus exists, however, we show the existence by construction through the following example of a two-payment mechanism.

Definition 9 (Two-choice, contingent-menu mechanism, $\text{CMM}(q)$). This is a DRM. Collect reported type $\hat{F}_i$ from each agent, and compute corresponding utility $\hat{u}_i(z, y)$ for penalty z and base payment y . Define a set of payment menus, $\Pi = \{\pi_\alpha\}_{\alpha \in \mathbb{R}^+}$ where each menu consists of two choices $\pi_\alpha = \{(q\alpha, 0), (0, \alpha)\}$, for parameter $q > 0$. Given a menu π_α , define $\hat{u}_i(\pi_\alpha) \triangleq \max_{(z, y) \in \pi_\alpha} \hat{u}_i(z, y)$. Let the *zero-crossing menu* for agent i , α_i , be the maximum α such that $\hat{u}_i(\pi_\alpha) \geq 0$.⁷

- Allocate the resource in period zero to the agent with the highest zero-crossing menu.
- Consider allocated agent i^* , and the “second best” agent, $i' = \arg \max_{i \neq i^*} \alpha_i$. The allocated agent’s two-part payment is $\arg \max_{(z, y) \in \pi_{\alpha_{i'}}} \hat{u}_i(z, y)$.

A larger class of mechanisms which have the two-payment mechanism defined in Definition 9 as a special case is as follows.

Definition 10 (Two choice, contingent-menu mechanism, $\text{CMM}(q, \gamma_1, \gamma_2)$). Consider the CMM with the set of menus Π parameterized by non-negative real numbers $\Pi = \{\pi_\alpha\}_{\alpha \in \mathbb{R}^+}$ where each menu $\pi_\alpha \in \Pi$ contains two contingent payment pairs:

$$\pi_\alpha = \left\{ \left(\alpha, \frac{\gamma_1}{1-\gamma_1} \alpha \right), \left(q\alpha, \frac{\gamma_2}{1-\gamma_2} q\alpha \right) \right\}$$

The elements of each menu can be seen as “ γ -CSP prices residing on straight lines through the origin with different slopes. $\gamma = 1$ corresponds to SP and $\gamma = 0$ corresponds to CSP (see Appendix B for more discussions). Define total order $\succeq$ on Π as: $\pi_{\alpha_1} \succeq \pi_{\alpha_2}$ iff $\alpha_1 \geq \alpha_2$ (as real numbers). Note that the CMM defined in Definition 9 has $\gamma_1 = 1$ and $\gamma_2 = 0$.

We would next show that this set of payment sets satisfy the assumptions (B1)-(B3).

Proof.

1. Each menu π_α is a discrete set with two elements, thus is closed (B1).
2. Denote $z_1(\alpha) = \alpha$, $y_1(\alpha) = \frac{\gamma_1}{1-\gamma_1} \alpha$, $z_2(\alpha) = q\alpha$, $y_2(\alpha) = \frac{\gamma_2}{1-\gamma_2} q\alpha$. It’s easy to see that for any type u , $u(z_1(\alpha), y_1(\alpha))$ and $u(z_2(\alpha), y_2(\alpha))$ are both continuous and strictly decreasing with α , thus the maximum of the two

$$u(\pi_\alpha) = \max(u(z_1(\alpha), y_1(\alpha)), u(z_2(\alpha), y_2(\alpha)))$$

is also continuous and monotonically decreasing with respect to α . (B3).

3. From (A1)-(A3) we know that $u(\pi_0) > 0$ and $\lim_{\alpha \rightarrow \infty} u(\pi_\alpha) < 0$ for any type u , thus there is a unique zero-crossing. (B2).

⁷We can show that $\hat{u}_i(\pi_\alpha)$ is a continuous and decreasing function of α , thus for each type we can uniquely define α_i .

□

We show in the following example with detailed explanation and illustrations, and also provide examples to show that CMM could work better than CSP by charging a higher penalty, or worse than CSP for certain economies by 1) charging a lower penalty and 2) not allocating to the CSP winner.

Example 11 (CMM(2) v.s. CSP). We consider the following three economies.

- Economy 1: Agent 1: exponential model with $w_1 = 10$, $\lambda_1 = 0.08$. Agent 2: exponential model with $w_2 = 15$, $\lambda_2 = 0.025$. We can compute the bids under CSP and CMM(2) as $b_{1,\text{CSP}} = 3.78$, $b_{2,\text{CSP}} = 2.22$, $\alpha_1 = 2.72$ and $\alpha_2 = 1.41$, as shown in Figure 19.

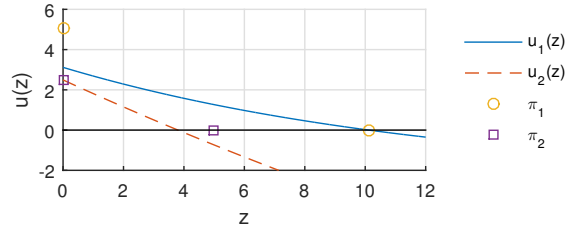


Figure 19: Expected utility curves and CMM payments for agents in economy 1, Example 11 where CMM(2) has better utilization than CSP.

We can see that agent 1 gets allocated the resource under both CSP and CMM(2). Under CSP, agent 1 pays the penalty at the zero-crossing point of agent 2. Under CMM(2), agent 1 faces two possible price in the menu determined by agent 2's report, marked as purple squares in Figure 19. Agent 1 gets a higher utility at the price on the horizontal axis thus is charged this price under CMM(2). The penalty is higher than that under CSP, thus utilization is improved (0.67 under CSP and 0.7 under CMM(2)).

- Economy 2: Agent 1: exponential model with $w_1 = 10$, $\lambda_1 = 0.08$. Agent 2: exponential model with $w_2 = 15$, $\lambda_2 = 0.025$. We can compute the bids as $b_{1,\text{CSP}} = 10.12$, $b_{2,\text{CSP}} = 3.80$, $\alpha_1 = 5.06$ and $\alpha_2 = 2.49$, as shown in Figure 19

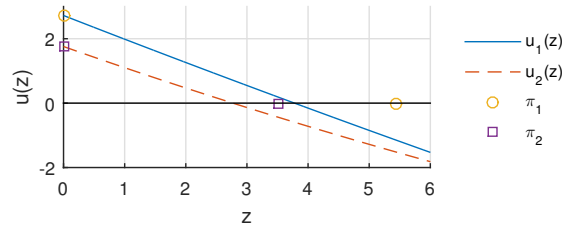


Figure 20: Expected utility curves and CMM payments for agents in economy 2, Example 11 where CMM(2) has worse utilization than CSP.

We can see that agent 1 gets allocated the resource under both mechanisms again. What is different is that in this economy, between the two prices in the menu π_2 , agent 1 gets a higher utility at the two part payment on the vertical axis, where there is no penalty! The utilization under CMM(2) for this economy is therefore worse than that of CSP, where agent 1 needs to pay the zero-crossing of the second agent as the penalty (compare 0.29 under CSP and 0.25 under CMM(2)).

• Economy 3: Agent 1: exponential model with $w_1 = 10$, $\lambda_1 = 0.08$. Agent 2: exponential model with $w_2 = 15$, $\lambda_2 = 0.025$. We can compute the bids under CSP and CMM(2) as $b_{1,\text{CSP}} = 9.44$, $b_{2,\text{CSP}} = 11.27$, $\alpha_1 = 6.80$ and $\alpha_2 = 5.64$, as shown in Figure 19

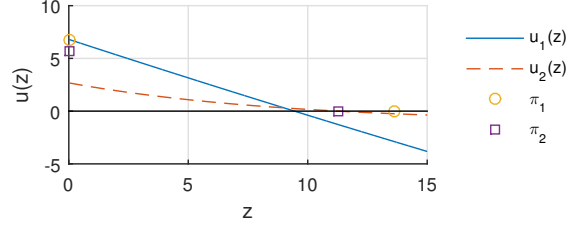


Figure 21: Expected utility curves and CMM payments for agents in economy 3, Example 11 where CMM(2) does not allocate to the CSP winner thus has way worse utilization than CSP.

We can see that in this economy, the resource is allocated to agent 2 under CSP but CMM(2) allocated the resource to agent 1, who has a smaller zero-crossing point but higher zero-crossing α . The utilization is therefore way lower than that under CSP (0.2592 under CMM(2) compared with 0.85 under CSP).

We can see that the reason why CMM is able to achieve a higher utilization than CSP is that it is able to charge the allocated agent in some economies a higher penalty than the second highest zero-crossing. This would not be individually rational for some other allocated agent, as discussed in the proof of Lemma 3. However, by offering some other payments to the allocated agent (and at least one of them must be on the frontier of the rest of the agents in the economy, as we will see), the IR issue is solved since any agent that must get allocated the resource but has negative utility at the high penalty can choose the other payment.

D Additional Simulation Results

D.1 Exponential Model

We first present the simulation results under exponential model, where we add in the contingent menu mechanisms (CMM) that are introduced in Appendix C.

Recall that under the exponential type model, $\mathbb{E}[V_i] = w_i - 1/\lambda_i$, where w_i is the value for using the resource and $1/\lambda_i$ is the expected opportunity cost. Assume $1/\lambda_i \sim U[0, 1/\Lambda]$, parameterized by $\Lambda > 0$ and that given λ_i , the value w_i is uniformly distributed $w_i \sim U[0, 1/\lambda_i]$. We set Λ , vary the number of agents from 2 to 15, and compute average utilization over 10,000 randomly generated economies. See Figure 22,⁸

The two-choice CMM(q) mechanism is tested with different q parameters, which determines the slope of the segment connecting the choices of two-part payments in each menu and how favorable the $(q\alpha, 0)$ payment is to an agent.

We see that CSP is much better than SP and the gap increases as the number of agents increases. With small q , CMM coincides with CSP since $(q\alpha, 0)$ is preferred by all agents. As q increases, the average utilization decreases, thus the payment menu is not helpful for utilization.

⁸The first-best value is the maximum possible utilization achievable given BB and IR, assuming that the mechanism has complete information about agent types. The “Opt. under P1-P5” value is the upper bound for any mechanism satisfying (P1)-(P5): the utilization at the zero-crossing point of the frontier, i.e. allocate to the CSP winner and charge her own zero-crossing as the penalty, which would not be truthful.

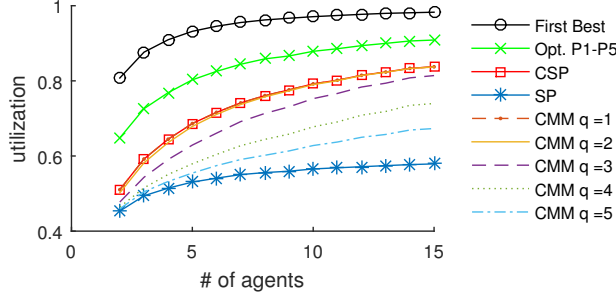


Figure 22: Average utilization under different mechanisms, along with the first-best and “opt under P1-P5” baselines.

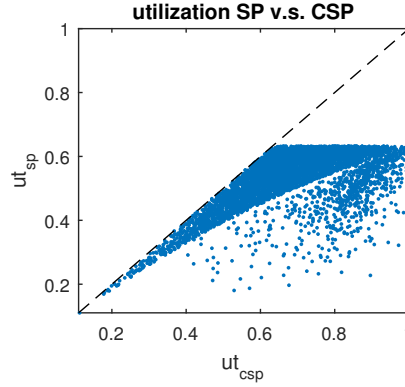


Figure 23: Utilization comparison between SP and CSP for exponential model. The dense “triangle” generally correspond to instances when SP allocates the resource to the same agent that gets allocated under CSP, and the rest of the points on the lower right side are instances in which SP and CSP allocate to different agents.

The utilization gap between CSP and the best possible utilization under (P1)-(P5) means that, the possibility remains of better mechanisms if we drop (P6) and (P7). We note, however, that the richer CMM mechanisms failed to close the gap. The utilization gap between the first-best (complete information) and the incomplete information baseline shows the cost of self-interest and private information on the ability to achieve a good coordination solution.

Fixing the number of agents to five, we zoom in and compare the utilization for the two mechanisms profile-by-profile, as the scatter plot in Figure 23. The horizontal coordinate for each point is the utilization under CSP for an economy, and the vertical axis is the utilization for the same economy under SP. As predicted by Theorem 4, the utilization under CSP is always higher.

Looking in some more depth at $\text{CMM}(q)$, we compare the utilization economy-by-economy between CSP and $\text{CMM}(q)$ for different q , as in Figure 24. Points above the diagonal line are instances where $\text{CMM}(q)$ has higher utilization, whereas points under the diagonal are cases when CSP works better.

All instances are with five agents. We can see that 1) when q is small, utilization under $\text{CMM}(q)$ is identical to that of CSP. This is because in any menu of payment $\pi_\alpha = \{(q\alpha, 0), (0, \alpha)\}$, the payment on horizontal axis $(q\alpha, 0)$ is more favorable for any agent type, when $q = 1$:

$$u(\alpha, 0) \geq u(0, \alpha), \forall u, \forall \alpha \geq 0$$

Therefore, the zero-crossing menu, allocation and payments are all defined by the $(q\alpha, 0)$ part of

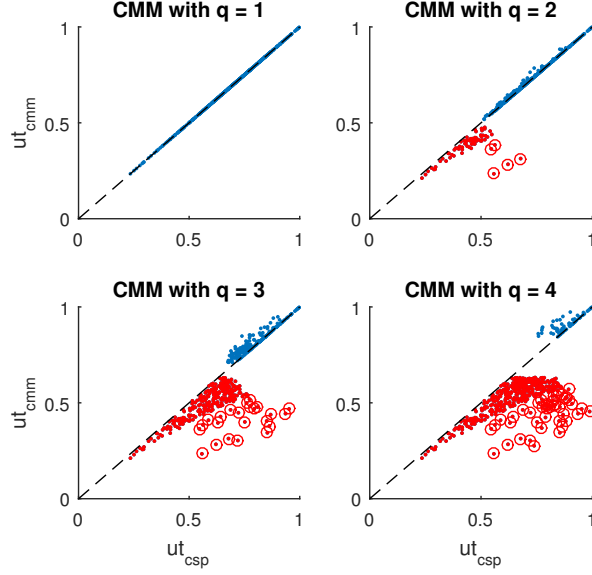


Figure 24: Utilization in CSP vs CMM(q) with different q for the exponential model. Blue dots: CMM is at least as high as CSP. Red dots: CMM is lower than CSP. Red circle: CMM allocates the resource to a different agent from the CSP winner.

the menus, thus the mechanism behaves exactly as CSP.

As q increases, we see some improvement in utilization when the allocated agent in CSP already has high utilization (thus has shallow utility curves) and prefers the $(q\alpha, 0)$ payment in CMM (which might be higher than the CSP penalty). Lower utilization CSP winners increasingly prefer the other payment $(0, \alpha)$, thus reducing the utilization because there is no penalty. This effect can be observed from Example 11 as well: the allocated agent prefers the $(q\alpha, 0)$ payment only if her probability of paying the penalty is relatively small, which means, her utilization is already high.

We will also observe that as q gets larger, there will be more instances when CMM allocates the resource to a different agent than the CSP winner (red circles). This would result in utilization a lot lower than that of CSP, which is the case for economy 3 in Example 11.

D.1.1 Uniform Model

We now introduce a new type model, the uniform model, where each agent's value is assumed to be uniformly distributed on some interval parameterized by $0 < a_2 < a_1$.

$$V_i \sim U[-a_{1,i}, a_{2,i}]$$

Assume that $a_{1,i}$'s are iid uniformly distributed on $[0, 10]$ and that given $a_{1,i}$, $a_{2,i}$ is uniformly distributed on $[0, a_{1,i}]$ (s.t. $\mathbb{E}[V_i] < 0$ and dominant strategies exist).

Average utilization over 10,000 iterations of various mechanisms and benchmarks are presented in Figure 25. Similar to the results that we see for exponential distribution, there is a gap between the upper bound of any mechanism satisfying (P1)-(P5), however, the simple family of CMM(q) mechanism is unable to close the gap.

Fix the number of agents to 10, we also examine the effect of reserve prices on utilization, shown in Figure 26. Increasing the reserve R from 0 to 10, the utilization of the CSP+R mechanism monotonically decreases, similar to the result that we have on exponential model in Section 6.

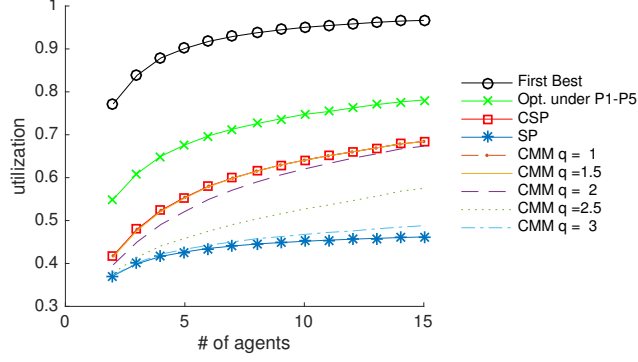


Figure 25: Utilization of CMM(q) mechanism, compared with SP, CSP and benchmarks for the uniform model

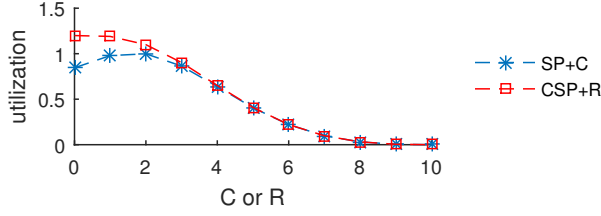


Figure 26: Utilization under CSP+R and SP+C.

D.1.2 (w_i, p_i) Model

We now present the average utilization of different mechanisms when agents' types are of the simple (w_i, p_i) model as introduced in Example 3. We use the most simple distribution, where w_i 's iid uniformly distributed on $[0, 10]$ and p_i 's are iid uniformly distributed on $[0, 1]$. Average utilization over 10,000 iterations as presented in Figure 27.

Observe that the utilization of CSP and the optimal benchmark under P1-P5 coincides. This is because the utilization of a (w_i, p_i) agent cannot be improved by penalties, thus the best any mechanism satisfying (P1)-(P5) is able to do is to select the agent at the zero-crossing of the frontier and reserve prices and CMM's would not be helpful.

D.2 Bids under Different Type Models

D.2.1 The $w_i p_i$ Model

The distribution under single value model:

$$V_i = \begin{cases} w_i & \text{w.p. } p_i \\ -\infty & \text{w.p. } 1 - p_i \end{cases}$$

for $w_i > 0$ and $p_i \in (0, 1)$. The CDF is given by:

$$F(v) = \begin{cases} 1 - p_i & \text{for } v < w_i \\ 1 & \text{for } v \geq w_i \end{cases}$$

Utility function

$$u(z, y) = \begin{cases} w_i p_i - (1 - p_i)z - y, & \text{for } z \geq -w_i \\ -z - y, & \text{for } z < -w_i \end{cases}$$

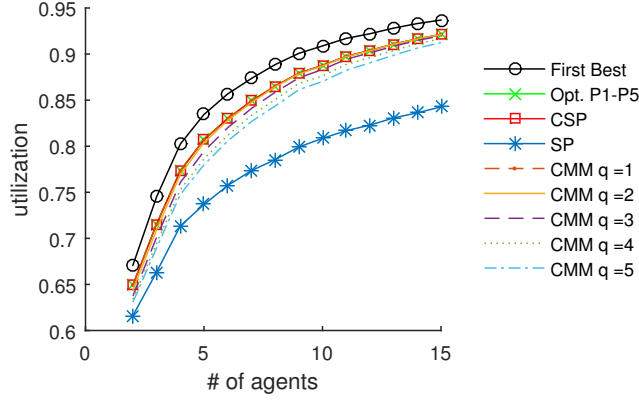


Figure 27: Utilization of CMM(q) mechanism, compared with SP, CSP and benchmarks for (w_i, p_i) Model

Zero-crossings

- Crossing with the vertical axis:

$$b_{SP} = y^* = -w_i p_i$$

- Crossing with the horizontal axis:

$$b_{CSP} = z^* = w_i p_i / (1 - p_i)$$

Other bids:

- SP+C:

$$b_{SP+C} = u(C, 0) = w_i p_i - (1 - p_i)C$$

- γ -CSP:

$$b_{\gamma CSP} = \frac{(1 - \gamma)w_i p_i}{1 - p_i + \gamma p_i}$$

- Fixed ratio $y = \beta z$:

$$b_{\beta} = \frac{w_i p_i}{1 + \beta - p_i}$$

Since the IR curve and BB curve never cross, the maximal z s.t. the mechanism is BB would be ∞ . The utilization would be

$$util(z, y) = p_i$$

regardless the prices.

D.2.2 The Uniform Model

Assume that the random variable for an agent:

$$V \sim U[-a_1, a_2]$$

where $-a_1 < 0 < a_2$ and $a_1 > |a_2|$. The CDF is given by

$$F(v) = \begin{cases} 0, & \text{for } v < -a_1 \\ (v - a_1)/(a_2 - a_1), & \text{for } -a_1 \leq v < a_2 \\ 1, & \text{for } v \geq a_2 \end{cases}$$

and the PDF:

$$f(v) = \begin{cases} 1/(a_1 + a_2), & \text{for } -a_1 \leq v \leq a_2 \\ 0, & \text{otherwise} \end{cases}$$

Utility function:

$$u(z, y) = \begin{cases} -z - y, & \text{for } z \leq -a_2 \\ \frac{z^2 - 2a_1z + a_2^2}{2(a_1 + a_2)} - y, & \text{for } -a_2 < z \leq a_1 \\ (a_2 - a_1)/2 - y, & \text{for } z > a_1 \end{cases}$$

Zero-Crossings:

- Zero-crossing with the vertical axis:

$$b_{SP} = y^* = u(0) = a_2^*/2(a_1 + a_2)$$

- Zero-crossing with the horizontal axis:

$$b_{CSP} = z^* = a_1 - \sqrt{a_1^2 - a_2^2}$$

Other bids:

- SP+C:

$$b_{SP+C} = u(C, 0)$$

- γ -CSP:

$$b_{\gamma CSP} = \frac{a_1 + \gamma a_2 - \sqrt{a_1^2 - a_2^2 + 2\gamma a_1 a_2 + 2\gamma a_2^2}}{1 - \gamma}$$

- Fixed ratio $y = \beta z$:

$$b_\beta = a_1 + \beta a_1 + \beta a_2 - \sqrt{-a_2^2 + (a_1 + \beta a_1 + \beta a_2)^2}$$

Utilization:

$$util(z) = \begin{cases} 0, & \text{for } v < -a_2 \\ (z + a_2)/(a_1 + a_2), & \text{for } -a_2 \leq v < a_1 \\ 1, & v \geq a_1 \end{cases}$$

Revenue:

$$rev(z, y) = z \cdot \frac{-z + a_1}{a_1 + a_2} + y$$

First best price:

$$z^{FB} = a_2, \quad y^{FB} = 2a_2(a_2 - a_1)$$

D.2.3 The Exponential Model

Consider the model where the value for using the resource equals to a fixed positive number w minus an exponentially distributed opportunity cost C with parameter λ :

$$V = w - C \text{ where } C \sim \text{Exp}(\lambda)$$

The CDF and PDF of the random variable is given by:

$$F(v) = \begin{cases} e^{\lambda(x-w)}, & \text{for } v \leq w \\ 1, & \text{for } v > w \end{cases}$$

$$f(v) = \begin{cases} \lambda e^{\lambda(v-w)}, & \text{for } v \leq w \\ 0, & \text{for } v > w \end{cases}$$

Utility function is for $z \geq -w$

$$\begin{aligned} u(z, y) &= \int_{-z}^w v \cdot \lambda \cdot e^{\lambda(v-w)} dv - z \cdot e^{-\lambda(z+w)} - y \\ &= w + \frac{1}{\lambda} \left(e^{-\lambda(w+z)} - 1 \right) - y \end{aligned}$$

and for $z < -w$

$$u(z, y) = z - y$$

Zero-crossings:

- Zero-crossing with the vertical axis:

$$b_{\text{SP}} = y^* = w + \frac{1}{\lambda} \left(e^{-\lambda w} - 1 \right)$$

- Zero-crossing with the horizontal axis:

$$b_{\text{CSP}} = z^* = -w - \frac{1}{\lambda} \log(1 - w\lambda)$$

Other bids:

- SP+C:

$$b_{\text{SP+C}}^* = w + \frac{1}{\lambda} \left(e^{-\lambda(w+C)} - 1 \right)$$

- γ -CSP

$$\begin{aligned} b_{\gamma\text{CSP}} &= \frac{1}{\gamma\lambda} \cdot [-1 + \gamma + w\lambda - w\gamma\lambda] + \frac{1}{\lambda} \cdot PL \left(-\frac{e^{-1+1/\gamma-w\lambda/\gamma}(-1+\gamma)}{\gamma} \right) \\ &= \frac{(w\lambda-1)(1-\gamma)}{\gamma\lambda} + \frac{1}{\lambda} PL \left(-\frac{e^{-1+1/\gamma-w\lambda/\gamma}(-1+\gamma)}{\gamma} \right) \end{aligned}$$

- Fixed ratio $y = \beta z$:

$$b_\beta = \frac{1}{\beta\lambda} \left[-1 + w\lambda + \beta W \left((e^{-w\lambda} - (w\lambda - 1)/\beta)/\beta \right) \right]$$

where $W(\cdot)$ is the Lambert W function, which is defined as the inverse of

$$z = we^w$$

Utilization:

$$util(z) = \begin{cases} 0, & \text{for } z < -w \\ 1 - e^{-\lambda(w+z)}, & \text{for } z \geq -w \end{cases}$$

Revenue:

$$rev(z, y) = z \cdot e^{-\lambda(w+z)} + y$$

First best:

$$z^{FB} = -\frac{1}{\lambda} - \frac{1}{\lambda} W \left(k, e^{-1+w\lambda}(-1 + w\lambda) \right)$$

where we need to take $k = -1$, since when $k = 0$, $W(k, e^{-1+w\lambda}(-1 + w\lambda)) = -1 + w\lambda \Rightarrow z^{FB} = -w$, which is not what we're looking for.

D.3 Distribution of Bids

D.3.1 (w_i, p_i) Model

Consider the most simple case when v_i and p_i are all independently uniformly distributed on $[0, 1]$.

Bids in SP For simplicity of notation take one bidder with $W \sim U[0, 1]$ and $P \sim U[0, 1]$. We know that in SP it's a dominant strategy to bid $B_{sp} = WP$ (also a random variable). We know from product distribution that for $b_{sp} \in [0, 1]$,

$$f(b_{sp}) = \int_{-\infty}^{\infty} f_W(w) f_P(b_{sp}/w) \frac{1}{|w|} dw = \int_0^1 1 \cdot \mathbb{1}\{b_{sp}/w \in [0, 1]\} \frac{1}{w} dw = \int_{b_{sp}}^1 \frac{1}{w} dw = \log \frac{1}{b_{sp}}$$

It's easy to check that this is a valid probability distribution.

Bids in CSP In CSP, the dominant strategy is to bid $B_{csp} = \frac{WP}{1-P}$. It is difficult to study the distribution of B_{csp} directly thus we consider the distribution of the log of the bid

$$\log B_{csp} = \log W + \log \frac{P}{1-P} = \log W + \log \left(\frac{1}{1-P} - 1 \right) \quad (8)$$

For $W \sim U[0, 1]$, denote $g(\cdot) = \log(\cdot)$ and $U = \log W = g(W)$. Since g is strictly increasing, the distribution of U would be

$$f_U(u) = \left| \frac{1}{g'(g^{-1}(u))} \right| f_W(g^{-1}(u)) = \frac{1}{1/e^u} \cdot \mathbb{1}\{e^u \in [0, 1]\} = \begin{cases} e^u, & u \leq 0 \\ 0, & u > 0 \end{cases}$$

Now we consider the second part. Since $P \sim U[0, 1]$, $1 - P$ is also uniformly distributed on $[0, 1]$. Denote $Q = \frac{1}{1-P}$, we know from inverse distribution that

$$f_Q(q) = \frac{1}{q^2} f_{1-P} \left(\frac{1}{q} \right) = \begin{cases} \frac{1}{q^2}, & q \geq 1 \\ 0, & q < 1 \end{cases}$$

Letting $R = \frac{1}{1-P} - 1 = Q - 1$, the distribution of R is

$$f_R(r) = f_Q(r+1) \left| \frac{dq}{dr} \right| = \begin{cases} \frac{1}{(r+1)^2}, & r \geq 0 \\ 0, & r < 0 \end{cases}$$

Denote $S = \log(R)$, its distribution is

$$f_S(s) = \frac{1}{1/e^s} f_R(e^s) = \frac{1}{1/e^s} \frac{1}{(e^s + 1)^2} = \frac{e^s}{(e^s + 1)^2}$$

Finally, $\log B_{csp} = U + S$ where U and S are independent, thus the distribution of B should be the convolution of the two pdfs:

$$f_{\log B_{csp}}(c) = \int_{-\infty}^{\infty} f_S(c-t) f_U(t) dt = \int_{-\infty}^0 \frac{e^{c-t}}{(e^{b-t} + 1)^2} e^t dt = e^c \left(\log(1 + e^{-c}) - \frac{1}{1 + e^c} \right)$$

This is actually not log-normal distributed, though it has a nice bell shape.

Now, the distribution of B_{csp} can be written as

$$f_{B_{csp}}(b) = \frac{1}{e^{\log b}} e^{\log b} \left(\log(1 + e^{-\log b}) - \frac{1}{1 + e^{\log b}} \right) = \log(1 + 1/b) - \frac{1}{1 + b}$$

D.3.2 Uniform Model

Consider the very simple uniform distribution s.t. the value $V \sim U[-a_1, a_2]$ where $0 < a_2 < a_1$. For simplicity, assume that:

$$\begin{aligned} a_1 &\sim U[0, 1] \\ a_2 &\sim U[0, a_1] \end{aligned}$$

We know that the bid under CSP:

$$b_{csp} = a_1 - \sqrt{a_1^2 - a_2^2} \triangleq b_{csp}(a_1, a_2)$$

It's easy to see that $b_{cap} < a_1$. Taking the partial derivatives, we have:

$$\begin{aligned} \frac{\partial}{\partial a_1} b_{csp}(a_1, a_2) &= 1 - \frac{a_1}{\sqrt{a_1^2 - a_2^2}} < 0 \\ \frac{\partial}{\partial a_2} b_{csp}(a_1, a_2) &= \frac{a_2}{\sqrt{a_1^2 - a_2^2}} > 0 \\ \frac{\partial^2}{\partial a_1 \partial a_2} b_{csp}(a_1, a_2) &= -a_1 a_2 (a_1^2 - a_2^2)^{-3/2} < 0 \end{aligned}$$

Thus for any $b < 1$, we know that

$$\begin{aligned}
\mathbb{P}[b_{csp} > b] &= \int_0^1 \int_0^{a_1} \frac{1}{a_1} \mathbb{1}\{b_{csp}(a_1, a_2) > b\} da_2 da_1 \\
&= \int_b^1 \frac{1}{a_1} \int_0^{a_1} \mathbb{1}\{a_1 - \sqrt{a_1^2 - a_2^2} > b\} da_2 da_1 \\
&= \int_b^1 \frac{1}{a_1} \int_0^{a_1} \mathbb{1}\{a_2 > \sqrt{2a_1b - b^2}\} da_2 da_1 \\
&= \int_b^1 \frac{1}{a_1} (a_1 - \sqrt{2a_1b - b^2}) da_1 \\
&= 1 - b - \int_b^1 \frac{\sqrt{2a_1b - b^2}}{a_1} da_1 \\
&= 1 - b - 2\sqrt{(2-b)b} + \frac{\pi}{2}b \\
&\quad - 2b \left(1 + \arctan\left(\frac{\sqrt{2-b}}{\sqrt{b}}\right) \right)
\end{aligned}$$

Thus the CDF of the bids under CSP is

$$\begin{aligned}
F(b) &= 1 - \mathbb{P}[b_{csp} > b] \\
&= b + 2\sqrt{(2-b)b} - \frac{\pi}{2}b + 2b \left(1 + \arctan\left(\frac{\sqrt{2-b}}{\sqrt{b}}\right) \right)
\end{aligned}$$

and the PDF of the bids is

$$f(b) = 1 - \frac{\sqrt{b}}{\sqrt{2-b}} - \frac{2(-1+b)}{\sqrt{(2-b)b}} + \frac{\pi}{2} - 2 \left(1 + \arctan\left(\frac{\sqrt{2-b}}{\sqrt{b}}\right) \right)$$

D.3.3 Exponential Model

Consider the case when $w - V$ (opportunity cost) is exponentially distributed with parameter λ and assume that the type distribution is

$$\begin{aligned}
\lambda^{-1} &\sim U[0, \Lambda^{-1}] \\
w &\sim U[0, \lambda^{-1}]
\end{aligned}$$

For simplicity of presentation, denote $\eta = \lambda^{-1} \sim U[0, \Lambda^{-1}]$. The bids under CSP would be

$$b_{csp} = -w - \frac{1}{\lambda} \log(1 - w\lambda) = -w - \eta \log(1 - w/\eta) \triangleq b_{csp}(\eta, w);$$

Taking the partial derivatives, we get

$$\begin{aligned}
\frac{\partial}{\partial \eta} b_{csp}(\eta, w) &= -\log(1 - w/\eta) - \frac{w/\eta}{1 - w/\eta} < 0 \\
\frac{\partial}{\partial w} b_{csp}(\eta, w) &= -1 + \frac{1}{1 - w/\eta} > 0
\end{aligned}$$

Thus for any $b > 0$, we know that

$$\begin{aligned} F(b) &= \mathbb{P}[b_{csp} < b] = \int_0^{\Lambda^{-1}} \left\{ \int_0^\eta \mathbb{1}\{b_{csp}(\eta, w) \leq b\} \frac{1}{\eta} dw \right\} \Lambda d\eta \\ &= \Lambda \int_0^\eta \frac{1}{\eta} \left\{ \int_0^{\lambda^{-1}} \mathbb{1}\{-w - \eta \log(1 - w/\eta) \leq b\} dw \right\} d\eta = \Lambda \int_0^{\Lambda^{-1}} \left(1 + W\left(-e^{-1-b/\eta}\right) \right) d\eta \end{aligned}$$

where $W(\cdot)$ is the Lambert W-function. This integration is hard to derive, however, we can write $W(\cdot)$ as an infinite series and compute the integration for each term separately.

First, note that $-b/\eta \leq 0$ for all possible $b \geq 0$ and $\eta \geq 0$, thus $-e^{-1-b/\eta} \geq -e^{-1}$. For $-e^{-1} < x \leq 0$, $W(x)$ can be written as

$$W(x) = \sum_{n=1}^{\infty} \frac{(-n)^{n-1}}{n!} x^n$$

Thus we can write the integrand as:

$$1 + W\left(-e^{-1-b/\eta}\right) = 1 + \sum_{n=1}^{\infty} \frac{(-n)^{n-1}}{n!} \left(-e^{-1-b/\eta}\right)^n = 1 - \sum_{n=1}^{\infty} \frac{n^{n-1}}{n!} e^{-n-nb/\eta}$$

For each term,

$$\int_0^{\Lambda^{-1}} e^{-n-nb/\eta} d\eta = e^{-n} \left(\Lambda^{-1} e^{-bn\Lambda} - bn\Gamma(0, bn\Lambda) \right)$$

where $\Gamma(a, z)$ is the upper incomplete Gamma function:

$$\Gamma(a, z) = \int_z^{\infty} t^{a-1} e^{-t} dt$$

Assume we are able to change the order of summation and integration, we get:

$$F(b) = \Lambda \int_0^{\Lambda^{-1}} \left(1 + W\left(-e^{-1-b/\eta}\right) \right) d\eta = 1 - \sum_{n=1}^{\infty} \frac{n^{n-1}}{n!} e^{-n} \left(e^{-bn\Lambda} - bn\Lambda\Gamma(0, bn\Lambda) \right)$$

Taking the derivative w.r.t. b , and assume we can change the order of summation and derivative, we get:

$$\begin{aligned} f(b) &= \frac{\partial}{\partial b} F(b) = - \sum_{n=1}^{\infty} \frac{n^{n-1}}{n!} e^{-n} \cdot \left(-n\Lambda e^{-bn\Lambda} - n\Lambda\Gamma(0, bn\Lambda) - bn\Lambda(-1)(bn\Lambda)^{-1} e^{-bn\Lambda} \right) \\ &= - \sum_{n=1}^{\infty} \frac{n^{n-1}}{n!} e^{-n} (-n\Lambda\Gamma(0, bn\Lambda)) = \Lambda \sum_{n=0}^{\infty} \frac{n^n}{n!} e^{-n} \Gamma(0, bn\Lambda) \end{aligned}$$

Evaluating $F(b)$ up to some finite n , we would get upper bounds on $F(b)$ as:

$$F^{(N)}(b) = 1 - \sum_{n=1}^N \frac{n^{n-1}}{n!} e^{-n} \left(e^{-bn\Lambda} - bn\Lambda\Gamma(0, bn\Lambda) \right)$$

Moreover, we can get a simple Lower bound on $F(b)$. Note that $W(x)$ is strictly concave, $W(-e^{-1}) = -1$ and $W(0) = 0$, we can lower bound $W(x)$ by:

$$W(x) \geq ex, \quad \forall x \in [-e^{-1}, 0]$$

Thus we can get:

$$F(b) \geq 1 - e^{-b\Lambda} + b\Lambda\Gamma(0, b\Lambda)$$