

Social Choice with Non Quasi-linear Utilities*

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Abstract

Without monetary payments, the Gibbard-Satterthwaite theorem proves that under mild requirements all truthful social choice mechanisms must be dictatorships. When payments are allowed, the Vickrey-Clarke-Groves (VCG) mechanism implements the value-maximizing choice, and has many other good properties: it is strategy-proof, onto, deterministic, individually rational, and does not make positive transfers to the agents. By Roberts' theorem, with three or more alternatives, the weighted VCG mechanisms are essentially unique for domains with quasi-linear utilities. The goal of this paper is to characterize domains of non-quasi-linear utilities where “reasonable” mechanisms (with VCG-like properties) exist. Our main result is a tight characterization of the maximal non quasi-linear utility domain, which we call the largest *parallel domain*. We extend Roberts' theorem to parallel domains, and use the generalized theorem to prove two impossibility results. First, any reasonable mechanism must be dictatorial when the utility domain is quasi-linear together with any single non-parallel type. Second, for richer utility domains that still differ very slightly from quasi-linearity, every strategy-proof, onto and deterministic mechanism must be a dictatorship.

1 Introduction

We study social choice mechanisms that aggregate individual preferences and select one among a finite set of alternatives. Our interest is in the existence of strategy-proof social choice mechanisms under general, non quasi-linear utility functions. In the classical voting problem without money, the seminal Gibbard-Satterthwaite theorem [Gibbard, 1973, Satterthwaite, 1975] states that if agents' preferences can be any ordering over the alternatives, the only deterministic, onto (i.e. every alternative can be selected) and strategy-proof mechanisms for three or more alternatives are dictatorial. On the other hand, with the introduction of monetary transfers and quasi-linear utilities, the Vickrey-Clarke-Groves (VCG) mechanism [Vickrey, 1961, Clarke, 1971, Groves, 1973] maximizes social welfare in dominant strategies, and can be generalized to implement any affine maximizer of agents' values [Roberts, 1979, Krishna and Perry, 1998].

However, quasi-linearity is a strong assumption, violated for example in domains with budget constraints and problems with lack of liquidity [Cramton, 1997] or with wealth effect and risk-aversion [Pratt, 1992]. Non quasi-linearity can also arise as a result of the timing of payments coupled with temporal preferences [Frederick et al., 2002], when payments are contingent on

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agents’ actions and the presence of payments affects decisions and thus the likelihood of contingencies [Ma et al., 2016b,c], and in the context of illiquid currencies such as points-based allocation schemes [Kash et al., 2007].

To the best of our knowledge, few papers considered social choice mechanisms with monetary transfers and non quasi-linear utilities.¹ The main result of this paper is a tight characterization of the maximal utility domain, which we name the *largest parallel domain*, where there exist non-dictatorial mechanisms that are strategy-proof, onto, deterministic, individually rational and satisfy *no subsidy* (i.e. no positive transfers from the mechanism to the agents). These properties are those of the VCG mechanism in quasi-linear utility domains. As a special case, we prove that for utility domains that contain all quasi-linear types but do not reside within the largest parallel domain, the only mechanisms satisfying the above conditions must be dictatorial. The proofs make use of a generalized Roberts’ theorem, which we extended to non quasi-linear utility domains with suitable properties. We also provide a negative result for a broader class of mechanisms: by allowing richer utility domains that still differ very slightly from quasi-linearity, we establish the impossibility of non-dictatorial mechanisms even without requiring individual rationality or no subsidy.

A key observation is that the critical property of agents’ utilities that enables non-dictatorial mechanisms is not the linear dependency on payments. We say the utility functions of an agent is of *parallel type* if for any two alternatives a and b , within the range of interest, no matter how much the agent is charged for b , to achieve the same utility, the additional amount she is willing to pay for a stays the same. Quasi-linear utility functions have this property, but there can also be non quasi-linear parallel types. Intuitively, a type being parallel requires that regardless of which alternative is selected, the agent’s marginal cost for money is the same as long as she has the same utility level—the trade-off with money depends on how happy the agent is, not how much she is paying. A domain where all types are parallel is called a *parallel domain*, and the largest parallel domain is the set of all parallel types.

The rest of the paper is organized as follows. After a brief discussion of related work, we provide in Section 2 a formal definition of parallel domains. We prove a positive result in Section 3, that within the parallel domains, the family of *generalized weighted VCG mechanisms* are strategy-proof, onto, deterministic, individually rational (IR), and satisfy no subsidy. In Section 4, we generalize Roberts’ Theorem [Roberts, 1979] to parallel domains, and prove that when the differences in agents’ willingness to pay for different alternatives are unrestricted, maximizers of affine functions of the willingness to pay are the only implementable choice rules amongst mechanisms with these properties. With this characterization, we prove in Section 5 our main result—that when agents have types outside of the parallel domain, the only mechanisms that are strategy-proof, onto, deterministic, individually rational, and satisfy no subsidy must be fixed-price dictatorships, i.e. there exists a dictator who chooses her favorite alternative given fixed prices associated with each alternative. We also develop a negative result for a broader class of mechanisms: by allowing utility domains that are slightly richer in their non quasi-linearity, we show that individual rationality and no subsidy can be relaxed, while the impossibility of non-dictatorial mechanisms can still be established. We give in Section 5.1, for example, an impossibility result for a *two-slopes domain*, when the utility function of each agent for each alternative is linear, with the slope taking one of two possible values. Proofs omitted from the body of the paper are provided in Appendix A.

¹A preliminary, short version of this paper [Ma et al., 2016a] proved a special case of the impossibility result for two agents and the richest non quasi-linear utility domain, with additional assumptions of unanimity and neutrality. The proof techniques are different, and this preliminary version does not make use of the generalized Roberts theorem.

1.1 Related Work

In social choice without monetary payments, the classical Gibbard-Satterthwaite theorem has been extended to more restricted preference domains, including the saturated domains [Kalai et al., 1979], linked domains [Aswal et al., 2003], circular domains [Sato, 2010], and weakly connected domains [Pramanik, 2015]. Domains for which there are positive results have also been extensively studied, see Black’s majority rule [Black, 1948], Moulin’s median voting schemes and generalizations [Moulin, 1980, Nehring and Puppe, 2007] and results on graphs with metric spaces [Barbera, 2001, Schummer and Vohra, 2002, Dokow et al., 2012].

For social choice with payments and quasi-linear utilities, Roberts [1979] showed that with three or more alternatives, when the values can take any real numbers, positive association of differences is necessary and sufficient for strategy-proof implementation, and that the only implementable choice rules are affine maximizers of agent values. Such choice rules can be implemented by weighted VCG mechanisms [Krishna and Perry, 1998]. Characterizations of strategy-proof implementations have also been developed for mechanism design problems in specific domains [Myerson, 1981, Rochet, 1987, Lavi et al., 2003, Bikhchandani et al., 2006, Saks and Yu, 2005].

One approach to mechanism design without quasi-linearity is to assume that the functional form of agent utility functions is known to the designer, for example auctions public budget constraints [Dobzinski et al., 2012], or auctions with known risk preferences [Maskin and Riley, 1984]. In contrast, we assume that the functional form of agent utility functions are private. For private non quasi-linear utilities, under suitable richness of type space, Kazumura et al. [2017] prove a “taxation principle” style characterization and a “revenue uniqueness” result of truthful mechanisms, and show various applications to problems other than social choice, e.g. single item allocation. The existence of truthful and non-dictatorial social choice mechanisms is not discussed.

For the assignment problem with unit demand, the minimum Walrasian-equilibrium mechanism is known to be truthful [Demange and Gale, 1985, Alaei et al., 2010, Aggarwal et al., 2009, Dütting et al., 2009, Morimoto and Serizawa, 2015], even for any general non-increasing utility function in payment. On the other hand, truthfulness cannot be achieved together with Pareto-efficiency for allocation problems in which agents may demand more than one unit of good, or when agents have multi-dimensional type spaces [Kazumura and Serizawa, 2016, Dobzinski et al., 2012, Baisa, 2016]. We do not impose Pareto-efficiency in proving our impossibility results. Randomized mechanisms for bilateral trade [Garratt and Pycia, 2015] and revenue-optimal auctions in very simple settings [Baisa, 2017, Che et al., 2012, Pai and Vohra, 2014] have also been studied in the context of private budget constraints. We focus here on social choice rather than assignment or allocation problems, settings for which there is more structure on agents’ preferences and also indifference toward outcomes where an agent’s own assignments are the same.

2 Preliminaries

Denote $N = \{1, 2, \dots, n\}$ as the set of *agents* and $A = \{a, b, \dots, m\}$ as the set of *alternatives*. A *social choice mechanism* accepts reports from agents as input, selects a single alternative $a^* \in A$, and may also determine payments. A mechanism is *onto* if for any alternative $a \in A$, there exists a preference profile for which a is selected. A mechanism is *dominant-strategy incentive compatible* (DSIC) if no agent can gain by reporting false preferences.

We allow monetary transfers, and the utility of an agent may depend both on the selected alternative and her assigned payment. Denote $u_{i,a}(z)$ as the utility of agent $i \in N$ if alternative $a \in A$ is selected and she needs to pay $z \in \mathbb{R}$. $u_i = (u_{i,a}, \dots, u_{i,m})$ determines agent i ’s *type* and is her private information. Denote $u = (u_1, \dots, u_n)$ as a *type profile*, and $u_{-i} = (u_1, \dots, u_{i-1}, u_{i+1}, \dots, u_n)$

as the type profile of agents except for agent i .

Denote the utility of alternative a to agent i at zero payment as $v_{i,a} \triangleq u_{i,a}(0)$, which we call the *value* of alternative a to agent i . Let $\bar{a}_i \in \arg \max_{a \in A} v_{i,a}$ and $\underline{a}_i \in \arg \min_{a \in A} v_{i,a}$ be a most and a least preferred alternative at zero payment. A utility profile u is *quasi-linear* if $u_{i,a}(z) = v_{i,a} - z$ for all $i \in N$, $a \in A$ and $z \in \mathbb{R}$. In this case, the values $\{v_{i,a}\}_{i \in N, a \in A}$ fully determine the type profile. Let the quasi-linear domain U_{QL} be the set of all quasi-linear types of a single agent where the $v_{i,a}$'s can take any value in $\mathbb{R}$, and let $\mathcal{U}_{QL} \triangleq \prod_{i=1}^n U_{QL}$ be the set of all quasi-linear type profiles.

We consider *non quasi-linear utilities* such that for all $i \in N$ and all $a \in A$,

(S1) $u_{i,a}(z)$ is continuous and strictly decreasing in z ,

(S2) $\lim_{z \rightarrow +\infty} u_{i,a}(z) < \min_{a' \in A} v_{i,a'}$.

Property (S1) guarantees that agents strictly prefer to make smaller payments. Property (S2) means that every agent prefers the worst alternative at zero payment to any alternative at some very large payment. Denote the general non quasi-linear utility domain U^0 as the set of all types of an agent satisfying (S1) and (S2), and let $\mathcal{U}^0 \triangleq \prod_{i=1}^n U^0$ be the general non quasi-linear utility domain for a set of n agents.

A *social choice mechanism* (x, t) on type domain $\mathcal{U} \subseteq \mathcal{U}^0$ is composed of a *choice rule* $x : \mathcal{U} \rightarrow A$ and a *payment rule* $t = (t_1, \dots, t_n) : \mathcal{U} \rightarrow \mathbb{R}^n$. Thus if the reported type profile is $\hat{u} \in \mathcal{U}$, the choice made is $x(\hat{u})$, and the utility of agent i is $u_{i,x(\hat{u})}(t_i(\hat{u}))$. A mechanism (x, t) is DSIC if and only if, for any agent $i \in N$, any type $u_i \in U_i$ of agent i , and any reported profile from other agents $\hat{u}_{-i} \in \mathcal{U}_{-i}$, agent i cannot gain by misreporting any type $\hat{u}_i \in U_i$:

$$u_{i,x(u_i, \hat{u}_{-i})}(t_i(u_i, \hat{u}_{-i})) \geq u_{i,x(\hat{u}_i, \hat{u}_{-i})}(t_i(\hat{u}_i, \hat{u}_{-i})). \quad (1)$$

A mechanism is *individually rational* (IR) if and only if, by truthfully participating in the mechanism, regardless of the reports made by the other agents, no agent can be worse off than having their worst alternative at zero payment selected and not making any payment.² That is, $\forall i \in N, \forall u_i \in U_i, \forall \hat{u}_{-i} \in \mathcal{U}_{-i}$,

$$u_{i,x(u_i, \hat{u}_{-i})}(t_i(u_i, \hat{u}_{-i})) \geq \min_{a \in A} v_{i,a}. \quad (2)$$

We are interested in mechanisms with the following set of properties.

- | | |
|--|---------------------------|
| P1. Dominant-strategy incentive compatible | P4. Individually rational |
| P2. Deterministic | |
| P3. Onto | P5. No subsidy |

Ontoness only requires that all alternatives will be selected given some type profile, but does not require all payment schedules are achievable. No subsidy requires that the mechanism does not make positive transfers to the agents.

Before continuing, we review a well-known characterization of deterministic DSIC mechanisms. We say that a mechanism is *agent-independent* if an agent's payment is independent of her report, conditioned on a particular alternative being selected; i.e. fixing the type profile of the rest of the agents u_{-i} , $\forall u_i, u'_i \in U_i$, $x(u_i, u_{-i}) = x(u'_i, u_{-i}) \Rightarrow t_i(u_i, u_{-i}) = t_i(u'_i, u_{-i})$. Given an agent-independent mechanism and any $u_{-i} \in \mathcal{U}_{-i}$, if there exists $u_i \in U_i$ s.t. $x(u_i, u_{-i}) = a$, let the *agent-independent price* be the payment i pays when a is selected: $t_{i,a}(u_{-i}) \triangleq t_i(u_i, u_{-i})$, which depends

²For a mechanism where IR is violated, an agent may benefit from not participating. See Section 5 for more discussions on *voluntary participation*. Assuming for all $i \in N$ and $a \in A$, $v_{i,a}$ may take any non-negative value, for DSIC mechanisms, this definition of IR is equivalent to requiring that the utility of any truthful agent is non-negative.

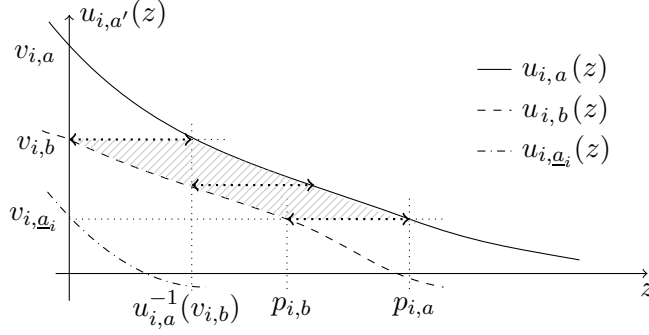


Figure 1: An example type in the parallel domain. All horizontal sections of the shaded area (e.g. the dotted arrows) are of the same length, i.e. utility curves are horizontal translations of each other within the range but need not be straight lines.

only on u_{-i} . Otherwise, if $x(u_i, u_{-i}) \neq a$ for all $u_i \in U_i$, let $t_{i,a}(u_{-i}) \triangleq +\infty$. An agent-independent mechanism is also *agent-maximizing* if given the agent-independent prices $\{t_{i,a}(u_{-i})\}_{i \in N, a \in A}$, the alternative selected by the mechanism maximizes the utilities of all agents simultaneously, i.e. $\forall u \in \mathcal{U}, \exists a^* \in A$ s.t. $a^* \in \arg \max_{a \in A} u_{i,a}(t_{i,a}(u_{-i}))$ for all $i \in N$, and $x(u) = a^*$. The properties of agent-independence and agent-maximization are necessary and sufficient for deterministic DSIC mechanisms with quasi-linear utilities [Vohra, 2011], and this equivalence can be easily generalized to general utilities that strictly decrease with payment.

A dictatorship in social-choice without money identifies an agent i^* as the dictator, and always selects her favorite alternative. We generalize this concept for social choice with money as follows:

Definition 1 (Fixed Price Dictatorship). Under a fixed price dictatorship, there exists a dictator $i^* \in N$ and fixed prices $\vec{z} \in \mathbb{R}^m$. Given any type profile $u \in \mathcal{U}$, one of the dictator's favorite alternatives under these prices is selected, i.e. $x(u) = a^* \in \arg \max_{a \in A} u_{i^*,a}(z_a)$, and the dictator pays $t_{i^*}(u) = z_{x(u)}$.

2.1 Parallel Domains

Given any type of an agent $u_i \in U^0$, for each alternative a , we define the *willingness to pay* $p_{i,a}$ as the payment for a at which the agent is indifferent between getting a at this payment, and getting her least preferred alternative $\underline{a}_i$ at zero payment:

$$p_{i,a} \triangleq u_{i,a}^{-1}(v_{i,\underline{a}_i}). \quad (3)$$

See Figure 1. $p_{i,a}$ is the maximum amount the agent can be charged if alternative a is selected, without violating IR. $p_{i,\underline{a}_i} = 0$ always holds, and (S1)-(S2) imply that for all $a \in A$, $p_{i,a}$ exists, and $0 \leq p_{i,a} < +\infty$.

Definition 2 (Parallel Domain). A utility domain $U_i \subset U^0$ for an agent is a *parallel domain* if for all $u_i \in U_i$,

$$u_{i,a}(z + (p_{i,a} - p_{i,b})) = u_{i,b}(z), \forall a, b \in A \text{ s.t. } v_{i,a} \geq v_{i,b}, \forall z \in [0, p_{i,b}]. \quad (4)$$

See Figure 1. We call a $u_i \in U^0$ a *parallel type* if (4) is satisfied. For a parallel type u_i , for $a, b \in A$ s.t. $v_{i,a} \geq v_{i,b}$, for any utility level $w \in [v_{i,\underline{a}_i}, v_{i,b}]$, we have

$$u_{i,a}^{-1}(w) - u_{i,b}^{-1}(w) = p_{i,a} - p_{i,b} = u_{i,a}^{-1}(v_{i,b}). \quad (5)$$

In words, the differences in the payments on a and b in order to achieve w is a constant that does not depend on w , i.e. the additional amount an agent is willing to pay for a over b does not depend on how much the agent is charged for b .³ For an agent with parallel type, her marginal cost for money is the same as long as she has the same utility level, regardless of which alternative is selected. Intuitively, the trade-off with money depends on how happy the agent is, not how much she is paying. For example, while paying for a better alternative, an agent with this kind of wealth effect would care less about money at the same payment amount than compared to a weaker alternative.

Denote $U_{\parallel} \subset U^0$ as the *largest parallel domain* (i.e. the set of all parallel types), and $\mathcal{U}_{\parallel} = \prod_{i=1}^n U_{\parallel}$. The quasi-linear domain $\mathcal{U}_{QL}$ is a parallel domain, where $p_{i,a} = v_{i,a} - v_{i,\underline{a}_i}$ for all $a \in A$ and $p_{i,a} - p_{i,b} = v_{i,a} - v_{i,b}$ for all $a, b \in A$. Another special case of the parallel domain is the *linear parallel domain*, where for every u_i , there exists $\alpha > 0$ s.t. $\forall a \in A$ and all $z \in \mathbb{R}$, $u_{i,a}(z) = v_{i,a} - \alpha z$ (so the quasi-linear domain is a special case when $\alpha = 1$). For these two domains, the “vertical” distances between the utility curves also stay the same (which need not be the case for a general parallel domain), and the utility functions are horizontal translations of each other everywhere.

A utility domain for an agent $U_i \subseteq U^0$ is said to have *unrestricted willingness to pay* if for any m -dimensional non-negative vector with at least one zero entry, there exists $u_i \in U_i$ s.t. the willingness to pay according to u_i is equal to this vector element-wise (at least one zero entry is required since an agent always has zero willingness to pay for $\underline{a}_i$). Formally, $\forall \lambda \in \mathbb{R}_{\geq 0}^m$ for which $\exists a \in A$ s.t. $\lambda_a = 0$, there exists $u_i \in U_i$ s.t. $p_{i,a} = \lambda_a$ for all $a \in A$.⁴ We call a parallel domain with unrestricted willingness to pay an *unrestricted parallel domain*. In particular, U_{QL} is an example of an unrestricted parallel domain. A utility domain $\mathcal{U} = \prod_{i=1}^n U_i$ is an *unrestricted parallel utility domain* if each of the U_i is unrestricted and parallel.

We now prove two lemmas.

Lemma 1. Let (x, t) be a DSIC and deterministic social choice mechanism on a utility domain $\mathcal{U} \subseteq U^0$ with unrestricted willingness to pay. The mechanism satisfies (P4) IR and (P5) No subsidy if and only if $\forall i \in N$ and $\forall u_{-i} \in \mathcal{U}_{-i}$, the agent-independent prices $\{t_{i,a}(u_{-i})\}_{a \in A}$ satisfy:

- (i) $t_{i,a}(u_{-i}) \geq 0$ for all alternatives $a \in A$,
- (ii) there exists an alternative $a \in A$ s.t. $t_{i,a}(u_{-i}) = 0$.

Thus, the agent-independent prices any agent faces under a mechanism satisfying (P1)-(P5) must be *standard*, i.e. the minimum price among all alternatives is zero. We leave the proof of the lemma to Appendix A.1.

Lemma 2. For any parallel type $u_i \in U_{\parallel}$ and any standard prices $\{t_{i,a}\}_{a \in A}$:

- (i) $\forall a, b \in A$ s.t. $0 \leq t_{i,a} \leq p_{i,a}$ and $0 \leq t_{i,b} \leq p_{i,b}$,

$$p_{i,a} - t_{i,a} \geq p_{i,b} - t_{i,b} \Leftrightarrow u_{i,a}(t_{i,a}) \geq u_{i,b}(t_{i,b}).$$

- (ii) $\arg \max_{a \in A} \{u_{i,a}(t_{i,a})\} = \arg \max_{a \in A} \{p_{i,a} - t_{i,a}\}$.

³There is no requirement on the shape of the utility functions below the utility level $v_{i,\underline{a}_i} = \min_{a \in A} v_{i,a}$, or where the payments are negative, since utilities in these ranges are irrelevant to the incentive properties of mechanisms that are IR and satisfy no subsidy. This also implies that the shape of the utility function of the least preferred alternative is irrelevant, and that when there are only two alternatives the general non quasi-linear domain U^0 is parallel.

⁴There are multiple (actually, infinite number of) types in U^0 with the same vector of willingness to pay, and we only require at least one of them to be included.

Proof. We first prove part (i). Assume w.l.o.g that $v_{i,a} \geq v_{i,b}$. We know from the monotonicity of $u_{i,a}$ and the definition of parallel domain (4) that

$$\begin{aligned} p_{i,a} - t_{i,a} \geq p_{i,b} - t_{i,b} &\Leftrightarrow t_{i,a} \leq t_{i,b} + p_{i,a} - p_{i,b} \\ \Leftrightarrow u_{i,a}(t_{i,a}) \geq u_{i,a}(t_{i,b} + (p_{i,a} - p_{i,b})) &\Leftrightarrow u_{i,a}(t_{i,a}) \geq u_{i,b}(t_{i,b}). \end{aligned}$$

For part (ii), observe that if the price for at least one of the alternatives is zero, the highest utility at the given prices among all alternatives $\max_{a \in A} \{u_{i,a}(t_{i,a})\}$ is at least $\min_{a \in A} u_{i,a}(0) = \min_{a \in A} v_{i,a}$. Therefore, for any alternative $a \in A$ s.t. $t_{i,a} > p_{i,a}$, the alternative cannot be agent-maximizing. Among the alternatives s.t. $t_{i,a} \leq p_{i,a}$, the agent-maximizing alternative(s) coincides with the maximizer(s) of $p_{i,a} - t_{i,a}$, according to part (i). $\square$

Thus, in a parallel domain, the agent-maximizing alternative given standard prices is the maximizer of the difference between the willingness to pay and the price: $p_{i,a} - t_{i,a}$. As a result, the willingness to pay serves similar roles as values in the quasi-linear domain, and it is this connection that enables us to generalize Roberts' theorem to unrestricted parallel domains.

3 The Generalized Weighted VCG Mechanism

We prove in this section a positive result, that in parallel domains, the generalized weighted VCG mechanisms implement in dominant strategy any affine maximizer of willingness to pay:

$$x(u) \in \arg \max_{a \in A} \left\{ \sum_{i=1}^n k_i p_{i,a} + C_a \right\}, \quad (6)$$

for non-negative weights $k_1, \dots, k_n \geq 0$, and real constants $\{C_a\}_{a \in A}$.

Definition 3 (Generalized Weighted VCG). The generalized weighted VCG mechanism, parametrized by non-negative weights $\{k_i\}_{i \in N}$ and real constants $\{C_a\}_{a \in A}$, collects a type profile $\hat{u} = (\hat{u}_1, \dots, \hat{u}_n)$ from agents, and computes the willingness to pay $\hat{p}_{i,a} = \hat{u}_{i,a}^{-1}(\min_{a' \in A} \hat{v}_{i,a'})$. It is defined as

- Choice rule: $x(\hat{u}) = a^*$, where $a^* \in \arg \max_{a \in A} \{\sum_{i \in N} k_i \hat{p}_{i,a} + C_a\}$, breaking ties arbitrarily, independent to agents' reports.
- Payment rule: $t_i(\hat{u}) = 0$ for $i \in N$ s.t. $k_i = 0$; for i s.t. $k_i \neq 0$:

$$t_i(\hat{u}) = \frac{1}{k_i} \left(\sum_{j \neq i} k_j \hat{p}_{j,a_{-i}^*} + C_{a_{-i}^*} - \sum_{j \neq i} k_j \hat{p}_{j,a^*} - C_{a^*} \right), \quad (7)$$

where $a_{-i}^* \in \arg \max_{a \in A} \{\sum_{j \neq i} k_j \hat{p}_{j,a} + C_a\}$.

Theorem 1. With type domain $\mathcal{U} \subseteq \mathcal{U}_{\parallel}$, any non-negative coefficients $\{k_i\}_{i \in N}$ and any real constants $\{C_a\}_{a \in A}$, the generalized weighted VCG mechanism is DISC, IR and does not make positive transfers to the agents.

Proof. We first consider an agent $i \in N$ s.t. $k_i > 0$. Given u_{-i} , we can check that for any u_i agent i reports s.t. $x(u_i, u_{-i}) = a$, agent i 's agent-independent payment would be

$$t_{i,a}(u_{-i}) = \frac{1}{k_i} \left(\sum_{j \neq i} k_j p_{j,a_{-i}^*} + C_{a_{-i}^*} - \sum_{j \neq i} k_j p_{j,a} - C_a \right). \quad (8)$$

For agent i s.t. $k_i = 0$, $\arg \max_{a \in A} \{\sum_{j \in N} k_j p_{j,a} + C_a\}$ and $\arg \max_{a \in A} \{\sum_{j \neq i} k_j p_{j,a} + C_a\}$ coincide. No matter what agent i reports, $a_{-i}^* = a^*$ is always selected and she does not pay anything, thus

$$t_{i,a_{-i}^*}(u_{-i}) = 0 \text{ and } t_{i,a}(u_{-i}) = +\infty \text{ for } a \neq a_{-i}^*. \quad (9)$$

Since a_{-i}^* is the maximizer of $\sum_{j \neq i} k_j p_{j,a} + C_a$, all agent-independent prices are non-negative. Moreover, a_{-i}^* has the minimum price among all alternatives, which is exactly zero: $\min_{a \in A} \{t_{i,a}(u_{-i})\} = t_{i,a_{-i}^*}(u_{-i}) = 0$. By Lemma 1, we know that the prices are standard, and the mechanism satisfies (P4) IR and (P5) No subsidy if it is DSIC. What is left to prove is choosing a^* at such agent-independent prices is agent-maximizing for all agents, which implies DSIC. From Lemma 2 we know that we only need to prove a^* is the maximizer of $p_{i,a} - t_{i,a}(u_{-i})$ for all agents. This is immediate for agents with $k_i = 0$. For an agent with $k_i > 0$, for any alternative $a \in A$, we have

$$\begin{aligned} & p_{i,a^*} - t_{i,a^*}(u_{-i}) - (p_{i,a} - t_{i,a}(u_{-i})) \\ &= p_{i,a^*} - \frac{1}{k_i} \left(\sum_{j \neq i} k_j p_{j,a_{-i}^*} + C_{a_{-i}^*} - \sum_{j \neq i} k_j p_{j,a^*} - C_{a^*} \right) \\ & \quad - p_{i,a} + \frac{1}{k_i} \left(\sum_{j \neq i} k_j p_{j,a_{-i}^*} + C_{a_{-i}^*} - \sum_{j \neq i} k_j p_{j,a} - C_a \right) \\ &= \frac{1}{k_i} \left(\sum_{j \in N} k_j p_{j,a^*} + C_{a^*} - \sum_{j \in N} k_j p_{j,a} - C_a \right) \geq 0, \end{aligned}$$

thus a^* indeed maximizes $p_{i,a} - t_{i,a}(u_{-i})$. $\square$

Note that when $\mathcal{U} \subseteq \mathcal{U}_{QL}$, we have $p_{i,a} = v_{i,a} - v_{i,a_i}$ for all $a \in A$, $i \in N$, and maximizing an affine function of the willingness to pay is equivalent to maximizing the same affine function of the values. Thus, this mechanism coincides with the weighted VCG mechanisms when utilities are quasi-linear. Ontoness is satisfied if $k_i > 0$ for at least one agent and when the utility domain is unrestricted.

4 Generalizing Roberts' Theorem

With quasi-linear utilities, Roberts [1979] showed that with three or more alternatives, if each agent's value for each alternative can be any real number, the choice rule of any social choice mechanism that is (P1) DISC, (P2) deterministic and (P3) onto must be a maximizer of some affine function of agents' values. With two additional conditions, (P4) IR and (P5) No subsidy, we generalize Roberts' theorem to the unrestricted parallel domains.

Theorem 2 (Roberts' Theorem on Parallel Domains). *With three or more alternatives and an unrestricted parallel utility domain $\mathcal{U}$, for every social choice mechanism satisfying (P1)-(P5), there exist non-negative weights $k_1, \dots, k_n$ (not all equal to zero) and real constants $C_1, \dots, C_m$ such that for all $u \in \mathcal{U}$,*

$$x(u) \in \arg \max_{a \in A} \left\{ \sum_{i=1}^n k_i p_{i,a} + C_a \right\}.$$

We prove in Lemma 3 that the *weak monotonicity* condition defined in terms of willingness to pay (which is equivalent to the W-Mon condition in terms of values [Bikhchandani et al., 2006] when utilities are quasi-linear, see Definition 4) is a necessary condition of incentive compatibility. Following the same steps as in the first proof of Roberts' theorem presented in Lavi et al. [2009] while treating willingness to pay as “values” in the proof, we conclude that affine maximizers of the willingness to pay are the only implementable choice rules (see Appendix A.2 for the details). The coefficients $\{k_i\}_{i \in N}$ cannot be all zero in order to satisfy (P3) Onto-ness.

Regarding the requirements on the utility domain: in the proof of Robert's theorem [Lavi et al., 2009], the values can take any real numbers, whereas the willingness to pay for a non quasi-linear type takes non-negative values and one of them has to be exactly zero. This does not prevent us from generalizing the proof, since what is necessary is that the differences in the values $v_{i,a} - v_{i,b}$ for all $a, b \in A$ can be any real numbers. We get this from the unrestricted parallel domain.

Definition 4 (Weak Monotonicity). Let $\mathcal{U} = \prod_{i=1}^n U_i \subseteq \mathcal{U}^0$ be a utility domain. A choice rule $x : \mathcal{U} \rightarrow A$ satisfies *weak monotonicity* (W-Mon) if for all $u_{-i} \in \mathcal{U}_{-i}$ and all $u_i, u'_i \in U_i$,

$$x(u_i, u_{-i}) = a, x(u'_i, u_{-i}) = b \Rightarrow p'_{i,b} - p_{i,b} \geq p'_{i,a} - p_{i,a}.$$

In words, W-Mon in a parallel domain means that if alternative a is selected under (u_i, u_{-i}) and alternative b is selected under (u'_i, u_{-i}) , the additional willingness to pay for b comparing with a according to u'_i , i.e. $p'_{i,b} - p'_{i,a}$, must be at least as big as the additional willingness to pay for b comparing with a according to u_i : $p_{i,b} - p_{i,a}$. This is a generalization of the W-Mon condition in terms of values as defined in [Bikhchandani et al., 2006], and the two are equivalent when utilities are quasi-linear, in which case $p_{i,a} - p_{i,b} = v_{i,a} - v_{i,b}$ holds for all $a, b \in A$.

Lemma 3. With any parallel utility domain $\mathcal{U} \subseteq \mathcal{U}_{\parallel}$, every social choice mechanism satisfying (P1), (P2), (P4) and (P5) must satisfy W-Mon.

Proof. Consider two types u_i, u'_i and a social choice mechanism (x, t) s.t. $x(u) = a$ and $x(u'_i, u_{-i}) = b$. We know from agent-maximization and Lemma 2 that facing prices $\{t_{i,a'}(u_{-i})\}_{a' \in A}$, alternative a must be a maximizer of $p_{i,a'} - t_{i,a'}(u_{-i})$ according to u_i , and alternative b must be a maximizer of $p'_{i,a'} - t_{i,a'}(u_{-i})$ according to u'_i . Thus, $p_{i,a} - t_{i,a}(u_{-i}) \geq p_{i,b} - t_{i,b}(u_{-i})$ and $p'_{i,b} - t_{i,b}(u_{-i}) \geq p'_{i,a} - t_{i,a}(u_{-i})$ must hold. Adding both sides of the two inequalities we get $p_{i,a} + p'_{i,b} \geq p_{i,b} + p'_{i,a} \Rightarrow p'_{i,b} - p_{i,b} \geq p'_{i,a} - p_{i,a}$. $\square$

5 Impossibility results

We now state the main result in this paper.

Theorem 3 (Dictatorship). *With at least three alternatives and a utility domain $\mathcal{U} = \prod_{i=1}^n U_i$ s.t.*

(C1) for each $i \in N$, U_i contains an unrestricted parallel domain,

(C2) for at least $n - 1$ agents, $U_i \not\subseteq U_{\parallel}$,

the only social choice mechanisms that satisfy (P1)-(P5) are fixed price dictatorships.

The quasi-linear domain is unrestricted and parallel, thus a special case of the theorem can be stated as: on any utility domain containing $\mathcal{U}_{QL}$, if the utility domains of at least $n - 1$ agents contain non-parallel types, the only mechanisms satisfying (P1)-(P5) are fixed price dictatorships. We provide here an outline of the proof and leave the full version to Appendix A.3.

Given any mechanism (x, t) with (P1)-(P5) on a utility domain $\mathcal{U}$ satisfying (C1) and (C2), we prove that the restriction of the mechanism on the parallel subspace $\tilde{\mathcal{U}} \triangleq \mathcal{U} \cap \mathcal{U}_{\parallel}$ must also satisfy (P1)-(P5). Theorem 2 then implies that on $\tilde{\mathcal{U}}$, the choice rule x must be the maximizer of some affine function of agents' willingness to pay. Fixing the choice rule, agent-independence and agent-maximization determine the agent-independent prices up to a constant (when there is no tie), and the requirement that prices being standard fully pins down $\{t_{i,a}(u_{-i})\}_{a \in A}$ as (8) and (9) for all $i \in N$ and any u_{-i} s.t. $u_{i'}$ is parallel for all $i' \neq i$. Theorem 2 requires that there exists at least one agent with a non-zero coefficient k_i . If the number of agents for whom $k_i > 0$ is more than one, $U_i \not\subset U_{\parallel}$ for at least one of them. Assume w.l.o.g. that $k_1, k_2 > 0$ and $\exists u_2^* \in U_2 \setminus U_{\parallel}$. Fixing some parallel type profile of the rest of the agents $u_{-1,-2}^*, \{t_{1,a}(u_2^*, u_{-1,-2}^*)\}_{a \in A}$ are not yet pinned down by the above mentioned characterization since u_2^* is not parallel. We prove that no agent-independent price $\{t_{1,a}(u_2^*, u_{-1,-2}^*)\}_{a \in A}$ guarantees that an alternative that is agent-maximizing for all agents always exists for all economies $(u_1, u_2^*, u_{-1,-2}^*)$ where $u_1 \in U_1$. This contradicts DSIC.

Now we know that there exists exactly one agent (say i^*) with a non-zero coefficient. This implies that when the reported profile is parallel (i.e. $u \in \mathcal{U} \cap \mathcal{U}_{\parallel}$), the outcome of the mechanism must be determined according to a fixed price dictatorship. By induction on the number of agents whose type is not parallel, we then prove that for any $u \in \mathcal{U}$, the outcome must also be determined by the same fixed price dictatorship.

On the Tightness of the Negative Result That the utility domain $\mathcal{U}$ contains an unrestricted parallel domain is necessary for Theorem 2. If the number of agents that has a type outside of the parallel domain is smaller than $n - 1$, there are at least two agents whose types are always parallel. The generalized weighted VCG mechanism with $k_i > 0$ only for these agents would satisfy all (P1)-(P5), and still would not be a dictatorship.

Regarding the properties (P1)-(P5), the mechanism being (P1) DSIC and (P2) deterministic are trivially necessary. (P3) ontoneess is required, since if some alternatives are never selected, the number of alternatives can be effectively reduced to two, in which case all types satisfying (S1) and (S2) are parallel, thus even for the most general $\mathcal{U}^0$, any generalized weighted VCG mechanism with coefficients not all zero satisfies (P1)-(P5).

The conditions on the utility domain (C1) and (C2) in the statement of Theorem 3 require only a small deviation from quasi-linearity or the parallel domain. As an example, with $n = 1$ and $m = 3$, the type domain $U_1 = U_{QL}$ and $U_2 = U_{QL} \cup \{u_2^*\}$ for any $u_2^* \notin U_{\parallel}$ (e.g. as illustrated in Figure 2) satisfies both (C1) and (C2). Assuming only (C1) and (C2), truthful mechanisms that violate one or both of (P4) and (P5) may still exist. For the above described utility domain, the mechanism which always adds 1 to agent 2's payment but otherwise functions exactly as a generalized weighted VCG mechanism satisfies (P1)-(P3) and (P5). More detailed discussions are provided in Appendix A.4, and we present an alternative impossibility result assuming only (P1)-(P3) in Section 5.1.

On the Fixed Price Dictatorship In a fixed price dictatorship, the dictator may still be asked to make some non-zero payments. It is clear that such a mechanism is DSIC, and when the prices that the dictator faces are standard, it is also IR and never pays the agents. In order to replace the fixed-price dictatorship in Theorem 3 with full dictatorship, i.e. the dictator chooses her favorite alternative free of charge, we can impose another condition, *voluntary participation* (VP), which means any agent can choose to walk away from the mechanism and accept the alternative decided by the rest of the agents without having to make any payment. If the dictator is charged a positive fixed price $z_a > 0$ for some alternative $a \in A$, and a is still the dictator's favorite choice under the fixed prices, then when a is selected by the sub-economy without the dictator (which should be the

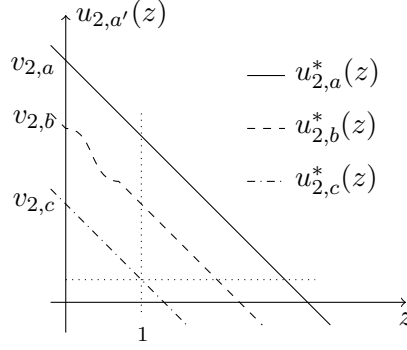


Figure 2: An example non-parallel type u_2^* of agent 2.

case for some u_{-i^*} given onto-ness), the dictator would have the incentive to walk away, in which case a will be selected and the dictator pays 0. This contradicts VP.

VP is stronger than IR. To see this, note that IR requires that for every agent at least one of the prices is weakly below the agent's willingness to pay. On the other hand, VP requires that the mechanism is well defined for any economy of $n-1$ agents and satisfy the same properties, moreover, each agent must face a zero-price for the alternative that would be selected in the sub-economy without her.

Regarding the payments from the rest of the agents—when the dictator strictly prefers a single alternative $a \in A$ (i.e. $\forall a' \neq a, u_{i^*,a'}(z_{a'}) < u_{i^*,a}(z_a)$), the rest of the agents do not make any payment: $t_i(u) = 0$ for all $i \neq i^*$. In the degenerate case where $|\arg \max_{a \in A} u_{i^*,a}(z_a)| > 1$, i.e. the dictator is indifferent toward multiple best alternatives, some or all the rest of the agents may be charged a non-zero payment to break ties among the dictator's favorite alternatives (e.g., we could run a generalized VCG mechanism between two alternatives that are tied from the dictator's perspective), and this may still satisfy (P1)-(P5). See Appendix A.3.

5.1 Relaxing IR and No Subsidy

We show in the rest of the section that with more richness in non quasi-linearity (e.g. with the linear domain with two slopes defined below), the dictatorship result remains given only (P1)-(P3).

Definition 5 (Linear Domain with Two Slopes). U_i is a linear domain with two slopes if there exists $\alpha_i, \beta_i > 0, \alpha_i \neq \beta_i$ s.t. for all $u_i \in U_i, \forall a \in A$, either $u_i(z) = v_{i,a} - \alpha_i z$ for all $z \in \mathbb{R}$ or $u_i(z) = v_{i,a} - \beta_i z$ for all $z \in \mathbb{R}$.

We say a linear domain with two slopes U_i is *unrestricted* if for each $i \in N$, the values $\{v_{i,a}\}$ can be any real numbers, and the slopes of utility functions for different alternatives can be any combination of α_i and β_i .

Theorem 4. *With at least three alternatives and a utility domain $\mathcal{U} = \prod_{i=1}^n U_i$ s.t. for each $i \in N$, U_i contains an unrestricted linear domain with two slopes, a social choice mechanism satisfying (P1)-(P3) must be a fixed price dictatorship.*

Intuitively, each U_i contains as a sub-domain an unrestricted linear parallel domain (e.g. the set of u_i s.t. $u_{i,a}(z) = v_{i,a} - \alpha_i z$ for all $z \in \mathbb{R}$ and all $a \in A$), which is a special case of a *strictly parallel domain*. For strictly parallel domains, with only (P1)-(P3), we can generalize Roberts' Theorem, and still determine the agent-independent prices up to a constant. We then prove that

if more than one agent has a positive coefficient in the choice rule, the induced agent-independent prices will result in contradictions when an agent's utility domain consists of utility functions with mixed slopes. Then we can prove by induction that the only agent with a positive coefficient must be a fixed price dictator. See Appendix A.4.1 for the full proof.

The theorem still holds if $\alpha_i = \alpha$, $\beta_i = \beta$ for all i . Moreover, we would reach the same result if the α_i 's and β_i 's are known to the mechanism designer. Since the α_i 's and the β_i 's can be arbitrarily close, this theorem shows that very slight disturbances on the slopes of agents' utility functions is sufficient to rule out the existence of truthful non-dictatorial mechanisms.

6 Conclusions

The existence of truthful and non-dictatorial social choice mechanisms strongly depends on whether monetary transfers are allowed. The seminal Gibbard-Satterthwaite theorem proves that without payments, the only truthful and onto mechanisms are dictatorial, whereas for the (rather restrictive) quasi-linear utility domain, any affine maximizer of agent values can be implemented if payments are allowed. We study social choice with payments and general utilities, distinguish types being *parallel* as the central property of quasi-linearity for DSIC mechanisms to exist, generalize (with additional conditions IR and No subsidy) Roberts' theorem to parallel domains with unrestricted willingness to pay, and provide a tight characterization of the largest parallel domain. Within the largest parallel domain, the generalized weighted VCG mechanisms implement any affine maximizer of agents' willingness to pay, and satisfy DSIC, onto, deterministic, IR and do not make payments to agents. Adding any non-parallel type to an unrestricted parallel domain, the only mechanisms with the above properties are dictatorial. We also discuss utility domains that are richer in their non quasi-linearity but still deviate very slightly from the quasi-linear domain, for which individual rationality and no subsidy can be relaxed and the dictatorship result remains.

References

- Gagan Aggarwal, S Muthukrishnan, Dávid Pál, and Martin Pál. General auction mechanism for search advertising. In *Proceedings of the 18th international conference on World wide web*, pages 241–250. ACM, 2009.
- Saeed Alaei, K Jain, and A Malekian. Competitive Equilibria in Two Sided Matching Markets with Non-transferable Utilities. *arXiv:1006.4696*, 10(2):34–36, 2010. URL <http://arxiv.org/abs/1006.4696>.
- Navin Aswal, Shurojit Chatterji, and Arunava Sen. Dictatorial domains. *Economic Theory*, 22(1):45–62, 2003.
- Brian Baisa. Efficient multi-unit auctions for normal goods. *SSRN 2824921*, 2016.
- Brian Baisa. Auction design without quasilinear preferences. *Theoretical Economics*, 12(1):53–78, 2017.
- Salvador Barbera. An introduction to strategy-proof social choice functions. *Social Choice and Welfare*, 18(4):619–653, 2001.
- Sushil Bikhchandani, Shurojit Chatterji, Ron Lavi, Ahuva Mu’alem, Noam Nisan, and Arunava Sen. Weak monotonicity characterizes deterministic dominant-strategy implementation. *Econometrica*, 74(4):1109–1132, 2006.
- Duncan Black. On the rationale of group decision-making. *The Journal of Political Economy*, pages 23–34, 1948.
- Yeon-Koo Che, Ian Gale, and Jinwoo Kim. Assigning resources to budget-constrained agents. *The Review of Economic Studies*, page rds025, 2012.
- Edward H Clarke. Multipart pricing of public goods. *Public choice*, 11(1):17–33, 1971.
- Peter Cramton. The fcc spectrum auctions: An early assessment. *Journal of Economics & Management Strategy*, 6(3):431–495, 1997.
- Gabrielle Demange and David Gale. The strategy structure of two-sided matching markets. *Econometrica: Journal of the Econometric Society*, pages 873–888, 1985.
- Shahar Dobzinski, Ron Lavi, and Noam Nisan. Multi-unit auctions with budget limits. *Games and Economic Behavior*, 74(2):486–503, 2012.
- Elad Dokow, Michal Feldman, Reshef Meir, and Ilan Nehama. Mechanism design on discrete lines and cycles. In *Proceedings of the 13th ACM Conference on Electronic Commerce*, pages 423–440. ACM, 2012.
- Paul Dütting, Monika Henzinger, and Ingmar Weber. Bidder optimal assignments for general utilities. In *International Workshop on Internet and Network Economics*, pages 575–582. Springer, 2009.
- Shane Frederick, George Loewenstein, and T O’donoghue. Time discounting and time preference: A critical review. *Journal of economic literature*, 40(2):351–401, 2002. URL <http://www.jstor.org/stable/2698382>.

- Rod Garratt and Marek Pycia. Efficient bilateral trade. Technical report, Working paper, available at SSRN 2444279.[975], 2015.
- Allan Gibbard. Manipulation of voting schemes: a general result. *Econometrica: journal of the Econometric Society*, pages 587–601, 1973.
- Theodore Groves. Incentives in teams. *Econometrica: Journal of the Econometric Society*, pages 617–631, 1973.
- Ehud Kalai, Eitan Muller, and Mark A Satterthwaite. Social welfare functions when preferences are convex, strictly monotonic, and continuous. *Public Choice*, 34(1):87–97, 1979.
- Ian A Kash, Eric J Friedman, and Joseph Y Halpern. Optimizing scrip systems: Efficiency, crashes, hoarders, and altruists. In *Proceedings of the 8th ACM conference on Electronic commerce*, pages 305–315. ACM, 2007.
- Tomoya Kazumura and Shigehiro Serizawa. Efficiency and strategy-proofness in object assignment problems with multi-demand preferences. *Social Choice and Welfare*, 47(3):633–663, 2016.
- Tomoya Kazumura, Debasis Mishra, and Shigehiro Serizawa. Mechanism design without quasilinearity. *SSRN 2982971*, 2017.
- Vijay Krishna and Motty Perry. Efficient mechanism design. 1998.
- Ron Lavi, Ahuva Mu’Alem, and Noam Nisan. Towards a characterization of truthful combinatorial auctions. In *Foundations of Computer Science, 2003. Proceedings. 44th Annual IEEE Symposium on*, pages 574–583. IEEE, 2003.
- Ron Lavi, Ahuva Mualem, and Noam Nisan. Two simplified proofs for roberts theorem. *Social Choice and Welfare*, 32(3):407–423, 2009.
- Hongyao Ma, Reshef Meir, and David C. Parkes. Social Choice for Agents with General Utilities. In *Proceedings of the 25th International Joint Conference on Artificial Intelligence (IJCAI’16)*, 2016a.
- Hongyao Ma, Reshef Meir, David C. Parkes, and James Zou. Contingent payment mechanisms to maximize resource utilization. *arXiv preprint arXiv:1607.06511*, 2016b.
- Hongyao Ma, Valentin Robu, Na Li, and David C. Parkes. Incentivizing Reliability in Demand-Side Response. In *Proceedings of the 25th International Joint Conference on Artificial Intelligence (IJCAI’16)*, 2016c.
- Eric Maskin and John Riley. Optimal auctions with risk averse buyers. *Econometrica: Journal of the Econometric Society*, pages 1473–1518, 1984.
- Shuhei Morimoto and Shigehiro Serizawa. Strategy-proofness and efficiency with non-quasi-linear preferences: A characterization of minimum price walrasian rule. *Theoretical Economics*, 10(2): 445–487, 2015.
- Hervé Moulin. On strategy-proofness and single peakedness. *Public Choice*, 35(4):437–455, 1980.
- Roger B Myerson. Optimal auction design. *Mathematics of operations research*, 6(1):58–73, 1981.

- Klaus Nehring and Clemens Puppe. The structure of strategy-proof social choice - part i: General characterization and possibility results on median spaces. *Journal of Economic Theory*, 135(1): 269–305, 2007.
- Malles M Pai and Rakesh Vohra. Optimal auctions with financially constrained buyers. *Journal of Economic Theory*, 150:383–425, 2014.
- Anup Pramanik. Further results on dictatorial domains. *Social Choice and Welfare*, 45(2):379–398, 2015.
- John W Pratt. Risk aversion in the small and in the large. In *Foundations of Insurance Economics*, pages 83–98. Springer, 1992.
- Kevin Roberts. The characterization of implementable choice rules. *Aggregation and revelation of preferences*, 12(2):321–348, 1979.
- Jean-Charles Rochet. A necessary and sufficient condition for rationalizability in a quasi-linear context. *Journal of mathematical Economics*, 16(2):191–200, 1987.
- Michael Saks and Lan Yu. Weak monotonicity suffices for truthfulness on convex domains. In *Proceedings of the 6th ACM conference on Electronic commerce*, pages 286–293. ACM, 2005.
- Shin Sato. Circular domains. *Review of Economic Design*, 14(3-4):331–342, 2010.
- Mark Allen Satterthwaite. Strategy-proofness and arrow’s conditions: Existence and correspondence theorems for voting procedures and social welfare functions. *Journal of economic theory*, 10(2):187–217, 1975.
- James Schummer and Rakesh V Vohra. Strategy-proof location on a network. *Journal of Economic Theory*, 104(2):405–428, 2002.
- William Vickrey. Counterspeculation, auctions, and competitive sealed tenders. *The Journal of finance*, 16(1):8–37, 1961.
- Rakesh V Vohra. *Mechanism design: a linear programming approach*, volume 47. Cambridge University Press, 2011.

A Proofs

A.1 Proof of Lemma 1

Lemma 1. Let (x, t) be a DSIC and deterministic social choice mechanism on a utility domain $\mathcal{U} \subseteq \mathcal{U}^0$ with unrestricted willingness to pay. The mechanism satisfies (P4) IR and (P5) No subsidy if and only if $\forall i \in N$ and $\forall u_{-i} \in \mathcal{U}_{-i}$, the agent-independent prices $\{t_{i,a}(u_{-i})\}_{a \in A}$ satisfy:

- (i) $t_{i,a}(u_{-i}) \geq 0$ for all alternatives $a \in A$,
- (ii) there exists an alternative $a \in A$ s.t. $t_{i,a}(u_{-i}) = 0$.

Proof. We first prove parts (i) and (ii) given (P4) and (P5). Assume part (i) does not hold, that there exists u_{-i} and $a \in A$ s.t. $t_{i,a}(u_{-i}) < 0$. Consider u_1 s.t. $a = \bar{a}_i$. In any agent-maximizing mechanism, the agent is guaranteed utility at least $u_{i,a}(t_{i,a}(u_{-i})) > \max_{a' \in A} v_{i,a'}$, which is not possible without the mechanism making a positive payment to the agent, and this violates (P5). Now we only need to prove that it cannot be the case that $t_{i,a}(u_{-i}) > 0$ for all $a \in A$. This is obvious, since if otherwise, there exists $u_i \in U_i$ for whom $p_{i,a} < t_{i,a}(u_{-i})$ for all $a \in A$ thus $\max_{a \in A} u_{i,a}(t_{i,a}(u_{-i})) < \min_{a \in A} v_{i,a}$, in which case (P4) is violated.

We now prove the other direction. Given part (i), it is obvious that no matter which alternative is selected, the transfer from any agent to the mechanism is non-negative thus (P5) holds. Given (ii), we know that the agent's minimum possible utility under an agent-maximizing mechanism is $v_{i,a}$, which is at least $\min_{a' \in A} v_{i,a'}$, thus (P4) holds. $\square$

A.2 Proof of Theorem 2

Theorem 2 (Roberts' Theorem on Parallel Domains). *With three or more alternatives and an unrestricted parallel utility domain $\mathcal{U}$, for every social choice mechanism satisfying (P1)-(P5), there exist non-negative weights $k_1, \dots, k_n$ (not all equal to zero) and real constants $C_1, \dots, C_m$ such that for all $u \in \mathcal{U}$,*

$$x(u) \in \arg \max_{a \in A} \left\{ \sum_{i=1}^n k_i p_{i,a} + C_a \right\}.$$

We had proved in Lemma 3 that W-Mon is a necessary condition for (P1)-(P5) if the utility domain is unrestricted and parallel. We provide here a few more steps where the details differ slightly for values and willingness to pay, following the first proof presented in Lavi et al. [2009], however, the high level ideas are the same.

Definition 6 (Positive Association of Differences). A social choice function x on a parallel domain $\mathcal{U}$ satisfies *positive association of differences* (PAD) if: for all u and u' in $\mathcal{U}$, if $x(u) = a$ and $p'_{i,a} - p_{i,a} > p'_{i,b} - p_{i,b}$ for all $b \neq a$ and all $i \in N$, then it must be the case that $x(u') = a$, as well.

Lemma 4 (IC $\Rightarrow$ PAD). A social choice mechanism on an unrestricted parallel domain $\mathcal{U}$ with (P1), (P2), (P4) and (P5) satisfies PAD.

Proof. By Lemma 3, x must satisfy W-Mon. Let $u, u' \in \mathcal{U}$ be type profiles such that $x(u) = a$ and $p'_{i,a} - p_{i,a} > p'_{i,b} - p_{i,b}$ for all $b \neq a$ and all $i \in N$. We need to show that $x(u') = a$. Denote $u^{(\ell)} = (u'_1, \dots, u'_\ell, u_{\ell+1}, \dots, u_n)$. We know $x(u^{(0)}) = x(u) = a$. Assume $x(u^{(\ell-1)}) = a$ for some $\ell > 0$, we show by contradiction that $x(u^{(\ell)}) = a$ must also hold. By induction, this implies that $x(u') = x(u^{(n)}) = a$.

Assume that there exists $\ell \geq 0$ and $b \neq a$ s.t. $x(u^{(\ell-1)}) = a$ but $x(u^{(\ell)}) = b \neq a$. Since all players except player ℓ have the same type in $u^{(\ell-1)}$ and in $u^{(\ell)}$, we get by W-Mon that $p'_{\ell,b} - p_{\ell,b} \geq p'_{\ell,a} - p_{\ell,a}$, which contradicts the PAD assumption on u' and u . Thus, $x(u^{(\ell)}) = a$ must hold. This completes the proof of the induction step. $\square$

We now prove the result analogous to Claim 1 in Lavi et al. [2009].

Claim 1. Assume a choice rule x on an unrestricted parallel domain $\mathcal{U}$ satisfies PAD. Fix type profiles $u, u' \in \mathcal{U}$ s.t. $x(u') = a$. If $p'_{i,b} - p_{i,b} > p'_{i,a} - p_{i,a}$ holds for all $i \in N$ for some $b \in A$, then $x(u) \neq b$.

Proof. We follow the same proof as in Lavi et al. [2009], and prove this claim by contradiction. The construction differs slightly from the original proof since the willingness to pay is always normalized s.t. the an agent's smallest willingness to pay among all alternatives is zero.

Suppose by contradiction that $x(u) = b$. For each $i \in N$, denote $\delta_i \triangleq p'_{i,b} - p_{i,b} - (p'_{i,a} - p_{i,a})$. We know that $\delta_i > 0$ for all $i \in N$, and in addition, $p_{i,a} - p'_{i,a} - \delta_i/2 = p_{i,b} - p'_{i,b} + \delta_i/2 > p_{i,b} - p'_{i,b}$. Consider $\tilde{p}$ s.t. for all $i \in N$,

$$\begin{aligned} \tilde{p}_{i,a} &= p_{i,a} - \delta_i/2, \\ \tilde{p}_{i,b} &= p_{i,b}, \end{aligned}$$

and for $c \in A$ s.t. $c \neq a, b$,

$$\tilde{p}_{i,c} = \min\{p_{i,c}, p'_{i,c} + p_{i,a} - p'_{i,a}\} - \delta_i.$$

Then, let p'' be the normalized version of $\tilde{p}$, i.e.

$$p''_{i,c} \triangleq \tilde{p}_{i,c} - \min_{d \in A} \{\tilde{p}_{i,d}\}, \quad \forall i \in N, \quad \forall c \in A.$$

First, we observe that $\min_{c \in A} \{p''_{i,c}\} = 0$ holds for all $i \in N$, i.e. p'' is a valid set of willingness to pay for the agents. Since $\mathcal{U}$ is unrestricted, we can find $u'' \in \mathcal{U}$ where agents' willingness to pay is given by p'' . Second, observe that for all $i \in N$ and any pair of alternatives $c, d \in A$, $p''_{i,c} - p''_{i,d} = \tilde{p}_{i,c} - \tilde{p}_{i,d}$.

We now show that the PAD condition from $u \rightarrow u''$ results in $x(u'') = b$, whereas applying PAD condition from $u' \rightarrow u''$ results in $x(u'') = a$, thus a contraction.

To show $x(u'') = b$ must hold, observe that for all $i \in N$,

$$\begin{aligned} p''_{i,b} - p_{i,b} - (p''_{i,a} - p_{i,a}) &= \tilde{p}_{i,b} - \tilde{p}_{i,a} - (p_{i,b} - p_{i,a}) \\ &= p_{i,b} - p_{i,a} + \delta_i/2 - (p_{i,b} - p_{i,a}) = \delta_i/2 > 0, \end{aligned}$$

and that for all $c \neq a, b$,

$$\begin{aligned} p''_{i,b} - p_{i,b} - (p''_{i,c} - p_{i,c}) &= \tilde{p}_{i,b} - \tilde{p}_{i,c} - (p_{i,b} - p_{i,c}) \\ &\geq p_{i,b} - p_{i,c} + \delta_i - (p_{i,b} - p_{i,c}) = \delta_i > 0. \end{aligned}$$

From Lemma 4, we know that $x(u'') = b$ must hold. Similarly, to show $x(u'') = a$ must hold, we can check that for all $i \in N$,

$$\begin{aligned} p''_{i,b} - p'_{i,b} - (p''_{i,a} - p'_{i,a}) &= \tilde{p}_{i,b} - \tilde{p}_{i,a} - (p'_{i,b} - p'_{i,a}) \\ &= p_{i,b} - p_{i,a} - (p'_{i,b} - p'_{i,a}) + \delta_i/2 = -\delta_i/2 < 0, \end{aligned}$$

and that for all $c \neq a, b$,

$$\begin{aligned} p''_{i,c} - p'_{i,c} - (p''_{i,a} - p'_{i,a}) &= \tilde{p}_{i,c} - \tilde{p}_{i,a} - (p'_{i,c} - p'_{i,a}) \\ &\leq p'_{i,c} + p_{i,a} - p'_{i,a} - \delta_i - (p_{i,a} - \delta_i/2) - (p'_{i,c} - p'_{i,a}) = -\delta_i/2 < 0. \end{aligned}$$

Lemma 4 then implies $x(u'') = a$ must hold. $\square$

We now consider the same sets that are analyzed in Lavi et al. [2009] in the first proof for Roberts' theorem, however, we define these sets in terms of differences in willingness to pay instead of differences in values. Let $\mathcal{U}$ be an unrestricted parallel domain. For all tuples $(a, b) \in A \times A$ that $a \neq b$:

$$P(a, b) \triangleq \{\alpha \in \mathbb{R}^n \mid \exists u \in \mathcal{U} \text{ s.t. } p_a - p_b = \alpha \text{ and } x(u) = a\}$$

The two immediate properties of the sets under quasi-linearity also holds:

1. For every a and b , the set $P(a, b)$ is not empty, as long as the choice rule x is onto.
2. If $\alpha \in P(a, b)$, then for any positive $\delta \in \mathbb{R}^n$ s.t. $\delta_i > 0$ for all i , $\alpha + \delta \in P(a, b)$. To see this, note that $\alpha \in P(a, b)$ implies that there exists u s.t. $x(u) = a$ and $p_a - p_b = \alpha$. Now we look for another type profile $u' \in \mathcal{U}$ s.t. comparing with u , we are increasing p_a by δ , while keeping the other willingness to pay the same. We know from PAD that $x(u') = a$ must still hold, and in this case $p'_a - p'_b = \alpha + \delta \in P(a, b)$ as required.

We now prove the following claim, which is analogous to Claim 2 in Lavi et al. [2009]. Without otherwise specify, the claims in the rest of this section assumes that there are three or more alternatives, that the utility domain is parallel and unrestricted, and that the mechanism satisfies (P1)-(P5).

Claim 2. For every $\alpha, \epsilon \in \mathbb{R}^n$, $\epsilon > 0$:

1. $\alpha - \epsilon \in P(a, b) \Rightarrow -\alpha \notin P(b, a)$.
2. $\alpha \notin P(a, b) \Rightarrow -\alpha \in P(a, b)$.

Proof. For the first part, note that $\alpha - \epsilon \in P(a, b)$ implies that there exists $u' \in \mathcal{U}$ s.t. $p'_a - p'_b = \alpha - \epsilon$ and $x(u') = a$. Now let $u \in \mathcal{U}$ be any type profile s.t. $p_b - p_a = -\alpha$. We know that for all $i \in N$, $p_{i,a} - p_{i,b} = \alpha_i > \alpha_i - \epsilon = p'_{i,a} - p'_{i,b}$. This implies $p'_{i,b} - p_{i,b} > p'_{i,a} - p_{i,a}$ for all $i \in N$, and we get $x(u) \neq b$ by applying Claim 1. This shows that for all $u \in \mathcal{U}$ s.t. $p_b - p_a = -\alpha$, $x(u) \neq b$, therefore $-\alpha \notin P(b, a)$.

For the second part, for any $c \neq a, b$, take some $\beta_c \in P(a, c)$ and fix some $\epsilon > 0$. Choose any u s.t. $p_a - p_b = \alpha$ and $p_a - p_c = \beta_c + \epsilon$ for all $c \neq a, b$. Since $p_a - p_b = \alpha \notin P(a, b)$, we know that $x(u) \neq a$. For all alternatives $c \neq a, b$, from Claim 1 and the fact that $\exists u'$ s.t. $p'_a - p'_c = \beta_c$ and $x(u') = a$, we know that $x(u) = c$ cannot hold. It follows that $x(u) = b$, thus $\exists u$ s.t. $p_b - p_a = -\alpha$ s.t. $x(u) = b$, thus $-\alpha \in P(b, a)$. $\square$

Intuitively, part 1 means that if for some u s.t. $p_a - p_b = \alpha - \epsilon$, $x(u) = a$, it must be the case that for all u s.t. $p_a - p_b = \alpha$, $x(u) \neq b$. This is because if a is selected under $p_a - p_b = \alpha - \epsilon$, when the difference in the willingness to pay for a vs. b increases by ϵ thus becomes larger, a still dominates b thus b cannot be selected.

Part 2 means that if for all u s.t. $p_a - p_b = \alpha$, $x(u) \neq a$, then there exists some u s.t. $p_a - p_b = \alpha$ and $x(u) = b$. This can be proved by constructing a type s.t. the willingness to pay for all alternatives other than a and b in a way that they are all dominated by a so cannot be selected, leaving b to be the only alternative that can be selected.

Claim 3. For every $\alpha, \beta, \epsilon^{(\alpha)}, \epsilon^{(\beta)} \in \mathbb{R}^n, \epsilon^{(\alpha)}, \epsilon^{(\beta)} > 0$:

$$\alpha - \epsilon^{(\alpha)} \in P(a, b), \beta - \epsilon^{(\beta)} \in P(b, c) \Rightarrow \alpha + \beta - \frac{\epsilon^{(\alpha)} + \epsilon^{(\beta)}}{2} \in P(a, c).$$

Proof. Choose any $u \in \mathcal{U}$ s.t. $p_a - p_b = \alpha - \epsilon^{(\alpha)}/2$ and $p_b - p_c = \beta - \epsilon^{(\beta)}/2$. We know from Claim 1 that $x(u) \neq b$ and $x(u) \neq c$. If the total number of alternatives is more than three, then for all $d \neq a, b, c$, fix some $\delta^{(w)} \in P(a, d)$ and some $\epsilon \in \mathbb{R}^n, \epsilon > 0$, and let $p_a - p_d = \delta^{(w)} + \epsilon$. Again by Claim 1, we know that $x(u) \neq d$. Therefore $x(u) = a$ must hold, and therefore $\alpha + \beta - (\epsilon^{(\alpha)} + \epsilon^{(\beta)})/2 \in P(a, c)$. $\square$

With the same argument as in the proof for quasi-linear Roberts' theorem, we know that if $\vec{0} \in P(a, b)$ for all pairs $(a, b) \in A \times A$, then the interior of all $P(a, b)$ must be equal. However, $P(a, b)$ does not necessarily include $\vec{0}$ thus needs to be “shifted” to contain this point. We can similarly define

$$\gamma(x, y) = \inf\{q \in \mathbb{R} \mid q \cdot \vec{1} \in P(a, b)\}.$$

It is easy to see that the infimum exists. We first argue that the set is not empty. Given that the social choice rule is onto, we can find some $u \in \mathcal{U}$ s.t. $x(u) = a$. Let $q = \max_i \{p_{i,a} - p_{i,b}\}$ and find u' s.t. $p'_{i,j} = p_{i,j}$ for all i and $j \neq a$, and $p'_{i,a} = p_{i,b} + q$ for all i . We know from PAD that $x(u') = a$ must still hold, thus this shows that $q \cdot \vec{1} \in P(a, b)$ thus the set is non-empty. To show that the set of q 's is lower-bounded, observe that if this is not the case, the corresponding set $P(b, a)$ would be empty, which contradicts onto-ness.

The rest of the proof in Lavi et al. [2009] for Roberts' theorem for quasi-linear utility domains analyzes of the sets $P(a, b)$ given the above lemmas and claims. Since these analysis does not depend on the values or the utility functions, the same identical arguments follow through for the parallel domain. This completes the proof of the Robert's theorem on unrestricted parallel domains.

A.3 Proof of Theorem 3

Theorem 3 (Dictatorship). *With at least three alternatives and a utility domain $\mathcal{U} = \prod_{i=1}^n U_i$ s.t.*

(C1) for each $i \in N$, U_i contains an unrestricted parallel domain,

(C2) for at least $n - 1$ agents, $U_i \not\subset U_{\parallel}$,

the only social choice mechanisms that satisfy (P1)-(P5) are fixed price dictatorships.

Before proving the main theorem, we first prove two lemmas. The following first lemma provides a characterization of agent-independent prices for mechanisms that satisfy (P1)-(P5).

Lemma 5. Fix any social choice mechanism on an unrestricted parallel domain $\mathcal{U}$ with choice rule $x(u) \in \arg \max_{a \in A} \sum_{i \in N} k_i p_{i,a} + C_a, \forall u \in \mathcal{U}$. If the mechanism satisfies (P1)-(P5), the agent-independent prices are determined by:

- (i) for $i \in N$ s.t. $k_i > 0$, for any $u_{-i} \in \mathcal{U}_{-i}$, let $a_{-i}^* \in \arg \max_{a \in A} \sum_{j \neq i} k_j p_{j,a} + C_a$, we have $t_{i,a}(u_{-i}) = 1/k_i (\sum_{j \neq i} k_j p_{j,a_{-i}^*} + C_{a_{-i}^*} - \sum_{j \neq i} k_j p_{j,a} - C_a)$ for all $a \in A$.
- (ii) for $i \in N$ s.t. $k_i = 0$, for any $u_{-i} \in \mathcal{U}_{-i}$, s.t. there exists $a_{-i}^* \in A$ that satisfies $\sum_{j \neq i} k_j p_{j,a} + C_a < \sum_{j \neq i} k_j p_{j,a^*} + C_{a^*}$ for all $a \neq a_{-i}^*$, we have $t_{i,a_{-i}^*}(u_{-i}) = 0$ and $t_{i,a}(u_{-i}) = +\infty, \forall a \neq a_{-i}^*$.

In other words, for agents s.t. $k_i \neq 0$, and for agents s.t. $k_i = 0$ but when there is no tie in the choice rule, the agent-independent prices must be determined by (8) and (9). As a consequence, when there is no tie, the outcome of the mechanism must be the same as that under the generalized weighted VCG mechanism.

Proof. Fix the choice rule as $x(u) \in \arg \max_{a \in A} \sum_{i=1}^n k_i p_{i,a} + C_a$, $\forall u \in \mathcal{U}$ for some non-negative coefficients $\{k_i\}_{i \in N}$ and real constants $\{C_a\}_{a \in A}$.

Part (i): Consider any agent $i \in N$ s.t. $k_i > 0$. For all $u_{-i} \in \mathcal{U}_{-i}$, for any two alternatives $a, b \in A$, there exists $u_i \in U_i$ such that both $\sum_{j \in N} k_j p_{j,a} + C_a > \sum_{j \in N} k_j p_{j,b} + C_b$ and $\sum_{j \in N} k_j p_{j,b} + C_b > \sum_{j \in N} k_j p_{j,c} + C_c$ hold, since U_i is unrestricted and $k_i > 0$. We know from the choice rule that only alternatives a and b can be selected, and a is selected if

$$\sum_{j \in N} k_j p_{j,a} + C_a > \sum_{j \in N} k_j p_{j,b} + C_b \Leftrightarrow p_{i,a} - p_{i,b} > \frac{1}{k_i} \left(\sum_{j \neq i} k_j p_{j,b} + C_b - \sum_{j \neq i} k_j p_{j,a} - C_a \right). \quad (10)$$

From Lemma 2, we also know that agent-maximization requires that a cannot be selected if

$$p_{i,a} - t_{i,a}(u_{-i}) < p_{i,b} - t_{i,b}(u_{-i}) \Leftrightarrow p_{i,a} - p_{i,b} < t_{i,a}(u_{-i}) - t_{i,b}(u_{-i}). \quad (11)$$

If the differences in the agent-independent prices satisfies $t_{i,a}(u_{-i}) - t_{i,b}(u_{-i}) > \frac{1}{k_i} (\sum_{j \neq i} k_j p_{j,b} + C_b - \sum_{j \neq i} k_j p_{j,a} - C_a)$, there exists $u_i \in U_i$ s.t. $p_{i,a} - p_{i,b} \in (\frac{1}{k_i} (\sum_{j \neq i} k_j p_{j,b} + C_b - \sum_{j \neq i} k_j p_{j,a} - C_a), t_{i,a}(u_{-i}) - t_{i,b}(u_{-i}))$ in which case both (10) and (11) hold. This is a contradiction. Similarly, we can show that it cannot be the case that $t_{i,a}(u_{-i}) - t_{i,b}(u_{-i}) < \frac{1}{k_i} (\sum_{j \neq i} k_j p_{j,b} + C_b - \sum_{j \neq i} k_j p_{j,a} - C_a)$, thus the price difference must be

$$t_{i,a}(u_{-i}) - t_{i,b}(u_{-i}) = \frac{1}{k_i} \left(\sum_{j \neq i} k_j p_{j,b} + C_b - \sum_{j \neq i} k_j p_{j,a} - C_a \right). \quad (12)$$

Since the choice of a and b are arbitrary, all prices are pinned-down up to a constant, given that the differences between any pair of prices are determined. Now observe $t_{i,a}(u_{-i})$ is smaller if $\sum_{j \neq i} k_j p_{j,a} + C_a$ is larger. From Lemma 1 we know that all prices must be non-negative, and that one of the prices must be zero, thus $t_{i,a^*_{-i}}(u_{-i})$, the smallest of all, must be exactly zero. Therefore we get:

$$t_{i,a^*_{-i}}(u_{-i}) = 0 = \frac{1}{k_i} \left(\sum_{j \neq i} k_j p_{j,a^*_{-i}} + C_{a^*_{-i}} - \sum_{j \neq i} k_j p_{j,a^*_{-i}} - C_{a^*_{-i}} \right),$$

and

$$\begin{aligned} t_{i,a}(u_{-i}) &= t_{i,a^*_{-i}} + \frac{1}{k_i} \left(\sum_{j \neq i} k_j p_{j,a^*_{-i}} + C_{a^*_{-i}} - \sum_{j \neq i} k_j p_{j,a} - C_a \right) \\ &= \frac{1}{k_i} \left(\sum_{j \neq i} k_j p_{j,a^*_{-i}} + C_{a^*_{-i}} - \sum_{j \neq i} k_j p_{j,a} - C_a \right). \end{aligned}$$

Part (ii): Now consider $i \in N$ s.t. $k_i = 0$. For all $u_{-i} \in \mathcal{U}_{-i}$ such that $\exists a^*_{-i} \in A$ that satisfies $\forall a \neq a^*_{-i}, \sum_{j \neq i} k_j p_{j,a} + C_a < \sum_{j \neq i} k_j p_{j,a^*_{-i}} + C_{a^*_{-i}}$, we know that no type of agent i in the parallel

domain $u_i \in U_{\parallel}$ will result in $x(u_i, u_{-i}) = a$ for $a \neq a_{-i}^*$. Therefore alternative a cannot be the unique agent-maximizing alternative for any $u_i \in U_i$, which implies $t_{i,a}(u_{-i}) = +\infty$ must hold. Now we know from Lemma 1 that $t_{i,a^*}(u_{-i}) = 0$ must be true, since one of the prices must be zero. $\square$

Given a utility domain $\mathcal{U} = \prod_{i=1}^n U_i$, denote the parallel sub-domain of each agent $i \in N$ as $\tilde{U}_i \triangleq U_i \cap U_{\parallel}$, and let $\tilde{\mathcal{U}} \triangleq \mathcal{U} \cap \mathcal{U}_{\parallel}$ be the subspace of $\mathcal{U}$ containing all parallel type profiles. If the utility domain $\mathcal{U}$ satisfies (C1), i.e. when U_i contains an unrestricted parallel domain, then each of the $\tilde{U}_i$ is an unrestricted parallel domain, and $\tilde{\mathcal{U}}$ is also unrestricted. For any social choice mechanism (x, t) on $\mathcal{U}$, we show that its restriction on $\tilde{\mathcal{U}}$ must inherit its good properties.

Lemma 6. Fix any social choice mechanism (x, t) under (P1)-(P3) on a utility domain $\mathcal{U}$ that satisfies (C1). The restriction of (x, t) on the parallel subdomain $\tilde{\mathcal{U}} = \mathcal{U} \cap \mathcal{U}_{\parallel}$ also satisfies (P1)-(P3). Moreover, if (x, t) also satisfies (P4) and (P5), then (P4) and (P5) are also satisfied by its restriction on $\tilde{\mathcal{U}}$.

Proof. For any mechanism that satisfies (P1)-(P3), it is immediate that (P1) DSIC (P2) deterministic must also hold for its restriction on any subdomain. Similarly, for a mechanism with (P1)-(P5), its restriction on any subdomain must also satisfy (P4) IR and (P5) no subsidy. What is left to show is that any mechanism that with (P1)-(P3) on a domain $\mathcal{U}$ that satisfies condition (C1), its restriction on $\tilde{\mathcal{U}}$ must also be onto. Assume toward a contradiction, that there exists $a^* \in A$ s.t. $\forall u \in \tilde{\mathcal{U}}, x(u) \neq a^*$. We prove by induction that the following statement holds for all $\ell \leq n - 1$:

$$F_{\ell}: \forall i \in N, \forall u_{-i} \in \mathcal{U}_{-i} \text{ such that } |\{i' \in N \mid i' \neq i, u_{i'} \notin \tilde{U}_i\}| \leq \ell, \text{ we have } t_{i,a^*}(u_{-i}) = \infty.$$

This implies that a^* cannot be the agent-maximizing alternative for any agent under any type profile $u \in \mathcal{U}$, thus a^* cannot be selected, and this violates the onto-ness of (x, t) on $\mathcal{U}$.

We first prove F_0 . When $|\{i' \in N \mid i' \neq i, u_{i'} \notin \tilde{U}_i\}| \leq 0$, we know $u_{-i} \in \tilde{\mathcal{U}}_{-i}$. For any $i \in N$ and any $u_{-i} \in \tilde{\mathcal{U}}_{-i}$, assuming $t_{i,a^*}(u_{-i}) < \infty$, there exists $u_i \in \tilde{U}_i$ s.t. $p_{i,a^*} - t_{i,a^*}(u_{-i}) > p_{i,a} - t_{i,a}(u_{-i})$ for all $a \neq a^*$, given that $\tilde{U}_i$ is unrestricted. In this case, given the parallel profile (u_i, u_{-i}) , a^* is the unique agent-maximizing alternative for agent i thus has to be selected. This contradicts the assumption that $x(u) \neq a^*$ for all $u \in \tilde{\mathcal{U}}$, thus $t_{i,a^*}(u_{-i}) = \infty$ must hold.

Now assume $F_{\ell-1}$ holds for some ℓ s.t. $1 \leq \ell \leq n - 2$, we show that F_{ℓ} also holds. W.l.o.g., we consider agent 1 and some u_{-1} s.t. $u_i \in \tilde{U}_i$ for all $i \geq \ell + 2$. In this case, only $u_2, \dots, u_{\ell+1}$ can be non-parallel. Now consider agent 2, and any $u_1 \in \tilde{U}_1$, we know in $u_{-2} = (u_1, u_{-1,-2})$, $|\{i \in N \mid i \neq 2, u_i \notin \tilde{U}_i\}| \leq \ell - 1$. As a result, $F_{\ell-1}$ implies that $t_{2,a^*}(u_{-2}) = \infty$, thus alternative a^* cannot be agent-maximizing for agent 2 and therefore cannot be selected in the economy (u_1, u_{-1}) . Since this holds for any $u_1 \in \tilde{U}_1$, with the same arguments that we proved F_0 , we conclude $t_{1,a^*}(u_{-1}) = \infty$ must hold as well, since otherwise a^* would be the unique agent-maximizing alternative for some parallel type u_1 and this violates DSIC. This proves $F_{\ell-1} \Rightarrow F_{\ell}$, and therefore completes the proof of this lemma. $\square$

We are now ready to prove the main theorem.

Proof of Theorem 3. Recall that $\tilde{U}_i \triangleq U_i \cap U_{\parallel}$ for each $i \in N$, and $\tilde{\mathcal{U}} \triangleq \mathcal{U} \cap \mathcal{U}_{\parallel}$. For any social choice mechanism (x, t) on $\mathcal{U}$ that satisfies (P1)-(P5), Lemma 6 implies that its restriction on $\tilde{\mathcal{U}}$ must also satisfy (P1)-(P5). Theorem 2 then guarantees that there exist non-negative coefficients $\{k_i\}_{i \in N}$, not all of them zero, and real constants $\{C_a\}_{a \in A}$ s.t. $x(u) \in \arg \max_{a \in A} \sum_{i \in N} k_i p_{i,a} + C_a$ for all $u \in \tilde{\mathcal{U}}$. Although this does not immediately determine the outcome of the mechanism for any non-parallel type profile $u \in \mathcal{U} \setminus \tilde{\mathcal{U}}$, we do know from agent independence that for any agent i ,

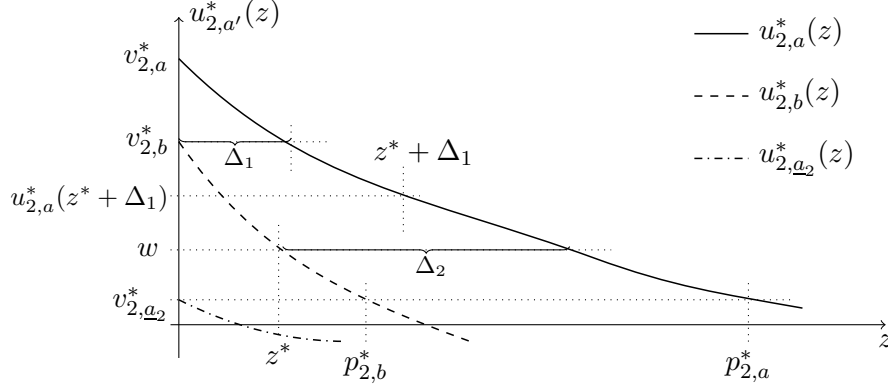


Figure 3: Illustration of $u_2^* \in U_2 \setminus \tilde{U}_2$, for Step 1 of the proof of Theorem 3.

and any parallel profile for the other agents $u_{-i} \in \tilde{U}_{-i}$, the agent-independent prices agent i faces $\{t_{i,a}(u_{-i})\}_{a \in A}$ must be characterized as in Lemma 5. We use this characterization and condition (C2) $U_i \not\subseteq U_{\parallel}$ for at least $n - 1$ agents to prove the dictatorship result in the following two steps:

- Step 1: the number of agents s.t. $k_i \neq 0$ is exactly one.
- Step 2: the only agent with $k_i \neq 0$ must be a fixed price dictator.

Step 1: We know from Theorem 2 that there exists at least one agent with $k_i > 0$. Assume towards a contradiction, that there exist at least two agents, which we name agent 1 and agent 2, for whom $k_1, k_2 > 0$. For at least one of them, say agent 2, $U_2 \subseteq U_{\parallel}$ does not hold due to condition (C2), thus there exists a non-parallel type $u_2^* \in U_2 \setminus \tilde{U}_2$. We prove that for some parallel profile for the rest of the agents $u_{-1,-2}^* \in \tilde{U}_{-1,-2}$, there do not exist agent-independent prices $\{t_{1,a}(u_2^*, u_{-1,-2}^*)\}_{a \in A}$ for agent 1, such that for any $u_1 \in \tilde{U}_1$, there exists an alternative that is agent-maximizing for all agents in economy $(u_1, u_2^*, u_{-1,-2}^*)$. This contradicts DSIC, therefore there is exactly one agent s.t. $k_i > 0$.

We assume w.l.o.g. that alternative a is one of agent 2's favorite alternatives at zero payment: $a \in \arg \max_{a' \in A} v_{2,a'}^*$ (where $v_{2,a}^* \triangleq u_{2,a}^*(0)$ for all $a \in A$). For any parallel type $u_2 \in U_{\parallel}$ s.t. $a \in \arg \max_{a' \in A} \{v_{2,a'}^*\}$, we know from the definition of the parallel domain that for any alternative $a' \in A$, $u_{2,a'}(z) = u_{2,a}(z + p_{2,a} - p_{2,a'}) = u_{2,a}(z + u_{2,a}^{-1}(v_{2,a'}^*))$ must hold for all $z \leq p_{2,a'}$. $u_2^* \notin U_{\parallel}$ implies that there exists some alternative $b \notin \arg \min_{a' \in A} v_{2,a'}^*$ and some price $z^* \in (0, p_{2,b}^*]$ s.t. $u_{2,b}^*(z^*) \neq u_{2,a}^*(z^* + (u_{2,a}^*)^{-1}(v_{2,b}^*))$. Let $w \triangleq u_{2,b}^*(z^*)$ and define $\Delta_1 \triangleq (u_{2,a}^*)^{-1}(v_{2,b}^*)$ and $\Delta_2 \triangleq (u_{2,a}^*)^{-1}(w) - z^*$. We assume $u_{2,b}^*(z^*) < u_{2,a}^*(z^* + \Delta_1)$, in which case $\Delta_2 > \Delta_1 \geq 0$, as illustrated in Figure 3. The other direction can be proved in the same way.

Fixing the type of the agents other than 1 and 2 to be parallel with zero willingness to pay on all alternatives, i.e. for all $i \geq 3$, let $u_i^* \in \tilde{U}_i$ be such that $p_{i,a'}^* = 0$ for all $a' \in A$. Such types exist since $\tilde{U}_i$ is unrestricted for each $i \in N$. Denote $u_{-1}^* = (u_2^*, u_3^*, \dots, u_n^*)$, and let ϵ be some small positive number s.t. $0 < \epsilon < (\Delta_2 - \Delta_1)/2$. We prove:

- Step 1.1: $t_{1,b}(u_{-1}^*) - t_{1,a}(u_{-1}^*) \leq k_2/k_1(\Delta_1 + \epsilon) + (C_a - C_b)/k_1$,
- Step 1.2: $t_{1,b}(u_{-1}^*) - t_{1,a}(u_{-1}^*) \geq k_2/k_1(\Delta_2 - \epsilon) + (C_a - C_b)/k_1$.

Since $k_2/k_1(\Delta_2 - \epsilon) + (C_a - C_b)/k_1 > k_2/k_1(\Delta_1 + \epsilon) + (C_a - C_b)/k_1$, we know that this is a contradiction, thus the number of agents for whom $k_i > 0$ cannot be more than one.

Step 1.1: Assume towards a contradiction that $t_{1,b}(u_{-1}^*) - t_{1,a}(u_{-1}^*) > k_2/k_1(\Delta_1 + \epsilon) + (C_a - C_b)/k_1$ and consider a parallel type $u_1 \in \tilde{U}_1$ of agent 1 with the following willingness to pay:

$$p_{1,c} = 0, \quad \forall c \neq a, b, \quad (13)$$

$$p_{1,b} = \max_{c \neq a, b} \left\{ \frac{k_2}{k_1} p_{2,c}^* + \frac{C_c - C_b}{k_1} \right\} + \delta, \quad (14)$$

$$p_{1,a} = p_{1,b} - \left(\frac{k_2}{k_1} (\Delta_1 + \epsilon/2) + \frac{C_a - C_b}{k_1} \right), \quad (15)$$

where δ is strictly positive, and large enough s.t. $p_{1,a}$ and $p_{1,b}$ as defined are both non-negative. Such u_1 is guaranteed to exist since $\tilde{U}_1$ is unrestricted. We know from (15) and the assumption $t_{1,b}(u_{-1}^*) - t_{1,a}(u_{-1}^*) > k_2/k_1(\Delta_1 + \epsilon) + (C_a - C_b)/k_1$ that:

$$\begin{aligned} & p_{1,a} - t_{1,a}(u_{-1}^*) - (p_{1,b} - t_{1,b}(u_{-1}^*)) \\ &= p_{1,a} - p_{1,b} + (t_{1,b}(u_{-1}^*) - t_{1,a}(u_{-1}^*)) \\ &> - \left(\frac{k_2}{k_1} (\Delta_1 + \epsilon/2) + \frac{C_a - C_b}{k_1} \right) + \left(\frac{k_2}{k_1} (\Delta_1 + \epsilon) + \frac{C_a - C_b}{k_1} \right) \\ &= \frac{k_2}{2k_1} \epsilon > 0, \end{aligned}$$

thus $p_{1,a} - t_{1,a}(u_{-1}^*) > p_{1,b} - t_{1,b}(u_{-1}^*)$. We conclude according to Lemma 2 that with prices $\{t_{1,a'}(u_{-1}^*)\}_{a' \in A}$, b cannot be an agent-maximizing alternative for agent 1.

We now prove that b is the only agent-maximizing alternative for agent 2, therefore no alternative can be agent-maximizing for both agents, leading to a contradiction to DSIC. First, by assumption, $p_{i,a'}^* = 0$ for all $i \neq 1, 2$ and all $a' \in A$, thus when the type profile of the rest of the economy is given by $(u_1, u_3^*, \dots, u_n^*)$, we have

$$\arg \max_{a' \in A} \left\{ k_1 p_{1,a'} + \sum_{i \geq 3} k_i p_{i,a'}^* + C_{a'} \right\} = \arg \max_{a' \in A} \{ k_1 p_{1,a'} + C_{a'} \}.$$

We can now check that $b \in \arg \max_{a' \in A} \{ k_1 p_{1,a'} + C_{a'} \}$. From (15) we know

$$\begin{aligned} & k_1 p_{1,b} + C_b - (k_1 p_{1,a} + C_a) = k_1 (p_{1,b} - p_{1,a}) + C_b - C_a \\ &= k_2 (\Delta_1 + \epsilon/2) + (C_a - C_b) - (C_a - C_b) = k_2 (\Delta_1 + \epsilon/2) > 0, \end{aligned}$$

thus $k_1 p_{1,b} + C_b > k_1 p_{1,a} + C_a$. Moreover, for any $c \neq a, b$, we know from (13) and (14) that

$$\begin{aligned} & k_1 p_{1,b} + C_b - (k_1 p_{1,c} + C_c) = k_1 (p_{1,b} - p_{1,c}) + C_b - C_c \\ &> k_1 \left(\frac{k_2}{k_1} p_{2,c}^* + \frac{C_c - C_b}{k_1} \right) + C_b - C_c = k_2 p_{2,c}^* \geq 0, \end{aligned}$$

therefore, $k_1 p_{1,b} + C_b > k_1 p_{1,c} + C_c$ holds for all $c \neq a, b$. Now we know $b \in \arg \max_{a' \in A} \{ k_1 p_{1,a'} + C_{a'} \}$ which implies $t_{2,b}(u_1, u_{-1,-2}^*) = 0$, according to Lemma 5. Thus we know the utility agent 2 gets from alternative b at the current price is: $u_{2,b}^*(t_{2,b}(u_1, u_{-1,-2}^*)) = u_{2,b}^*(0) = v_{2,b}^*$.

For alternative a , we know from (12) and (15) that

$$\begin{aligned} & t_{2,a}(u_1, u_{-1,-2}^*) = t_{2,b}(u_1, u_{-1,-2}^*) + \frac{1}{k_2} (k_1 (p_{1,b} - p_{1,a}) + C_b - C_a) \\ &= 0 + \frac{1}{k_2} (k_2 (\Delta_1 + \epsilon/2) + C_a - C_b + C_b - C_a) = \Delta_1 + \epsilon/2 > \Delta_1. \end{aligned}$$

Therefore, $u_{2,a}^*(t_{2,a}(u_1, u_{-1,-2}^*)) < u_{2,a}^*(\Delta_1) = v_{2,b}^*$. For all other alternatives $c \neq a, b$, we know from (12), (13) and (14) that

$$\begin{aligned} t_{2,c}(u_1, u_{-1,-2}^*) &= t_{2,b}(u_1, u_{-1,-2}^*) + \frac{1}{k_2} (k_1(p_{1,b} - p_{1,c}) + C_b - C_c) \\ &> 0 + \frac{1}{k_2} \left(k_1 \left(\frac{k_2}{k_1} p_{2,c}^* + \frac{C_c - C_b}{k_1} \right) + C_b - C_c \right) = p_{2,c}^*. \end{aligned}$$

Therefore, $u_{2,c}^*(t_{2,c}(u_1, u_{-1,-2}^*)) < \min_{a' \in A} v_{2,a'}^* \leq v_{2,b}^*$ for all $c \neq a, b$. This proves that $\{b\} = \arg \max_{a' \in A} u_{2,a'}^*(t_{2,a'}(u_1, u_{-1,-2}^*))$, thus completes the proof of part Step 1.1.

Step 1.2: Assume for contradiction that $t_{1,b}(u_{-1}^*) - t_{1,a}(u_{-1}^*) < k_2/k_1(\Delta_2 - \epsilon) + (C_a - C_b)/k_1$. As discussed above, b cannot be the least preferred alternative at zero price according to u_2^* , thus we assume w.l.o.g. $m \in \arg \min_{a' \in A} v_{2,a'}^*$. Consider the type $u_1 \in \tilde{U}_1$ s.t.

$$\begin{aligned} p_{1,c} &= 0, \quad \forall c \neq a, b, m, \\ p_{1,a} &= \max_{c \neq a, b, m} \left\{ \frac{k_2}{k_1} p_{2,c}^* + \frac{C_c - C_a}{k_1} \right\} + \delta, \\ p_{1,b} &= p_{1,a} + \left(\frac{k_2}{k_1} (\Delta_2 - \epsilon/2) + \frac{C_a - C_b}{k_1} \right), \\ p_{1,m} &= p_{1,b} + \frac{1}{k_1} (k_2 z^* + C_b - C_m), \end{aligned}$$

where δ is some non-negative number such that $\min\{p_{1,a}, p_{1,b}, p_{1,m}\} \geq 0$.⁵ Similar to the proof of part Step 1.1, for u_1 as constructed, we can show that a cannot be the agent-maximizing alternative for agent 1 given prices $\{t_{1,a'}(u_{-1}^*)\}_{a' \in A}$ since $p_{1,b} - t_{1,b}(u_{-1}^*) > p_{1,a} - t_{1,a}(u_{-1}^*)$ thus a is not the maximizer of $p_{1,a'} - t_{1,a'}(u_{-1}^*)$. Moreover, we can prove that a is the unique agent-maximizing alternative for agent 2, by showing:

- 1) $m \in \arg \max_{a' \in A} \{k_1 p_{1,a'} + C_{a'}\}$ which implies $t_{2,m}(u_1, u_{-1,-2}^*) = 0$, $t_{2,a'}(u_1, u_{-1,-2}^*) = 1/k_2(k_1 p_{1,m} + C_m - k_1 p_{1,a'} - C_{a'})$ for all $a' \in A$, and $u_{2,m}^*(t_{2,m}(u_1, u_{-1,-2}^*)) = \min_{a' \in A} v_{2,a'}^* \leq w$.
- 2) $t_{2,b}(u_1, u_{-1,-2}^*) = z^*$ thus $u_{2,b}^*(t_{2,b}(u_1, u_{-1,-2}^*)) = u_{2,b}^*(z^*) = w$,
- 3) $t_{2,c}(u_1, u_{-1,-2}^*) > p_{2,c}^*$, thus $u_{2,c}^*(t_{2,c}(u_1, u_{-1,-2}^*)) < \min_{a' \in A} v_{2,a'}^* \leq w$ for all $c \neq a, b, m$, and
- 4) $t_{2,a}(u_1, u_{-1,-2}^*) = \Delta_2 + z^* - \epsilon/2 < \Delta_2 + z^*$, therefore $u_{2,a}^*(t_{2,a}(u_1, u_{-1,-2}^*)) > u_{2,a}^*(\Delta_2 + z^*) = w \geq \max_{a' \neq a} \{u_{2,a'}^*(t_{2,a'}(u_1, u_{-1,-2}^*))\}$.

This shows that no alternative is agent-maximizing for both agents 1 and 2, and completes the proof of this Step 1.2, and also Step 1.

Step 2: The mechanism must be a fixed price dictatorship.

So far, we have proved that for any mechanism (x, t) satisfying (P1)-(P5), its restriction on the parallel subdomain $\tilde{U}$ must be an affine maximizer of willingness to pay with coefficients $\{k_i\}_{i \in N}$ and constants $\{C_a\}_{a \in A}$, where $k_i > 0$ for exactly one agent. Let's name her agent 1 and let $\bar{z}$ be a vector of fixed prices in $\mathbb{R}_{\geq 0}^m$ s.t.

$$z_a \triangleq \frac{1}{k_1} \left(\max_{a' \in A} \{C_{a'}\} - C_a \right), \quad \forall a \in A. \quad (16)$$

⁵If the number of alternatives is exactly 3, then we may set $p_{1,a} = \delta$ where $\delta \in \mathbb{R}$ guarantees $\min\{p_{1,a}, p_{1,b}, p_{1,m}\} = 0$, so that the smallest willingness to pay among all alternatives is zero. The rest of the proof remains the same.

To show that agent 1 is a fixed price dictator, i.e. $x(u) \in \arg \max_{a \in A} u_{1,a}(z_a)$ and $t_1(u) = z_{x(u)}$ for all $u \in \mathcal{U}$, agent-maximization implies that it is sufficient to show for all $u_{-1} \in \mathcal{U}_{-1}$, $t_{1,a}(u_{-1}) = z_a$ holds. We prove this by induction on the number of agents whose types are not parallel in the profile u_{-1} . For any $\ell = 0, 1, \dots, n-1$, let the induction statements be

G_ℓ : For all $u_{-1} \in \mathcal{U}_{-1}$ s.t. $|\{i \in N | i \neq 1, u_i \notin U_\parallel\}| \leq \ell$, $t_{1,a}(u_{-1}) = z_a$ holds for all $a \in A$.

H_ℓ : For all $i \neq 1$, for all $u_{-i} \in \mathcal{U}_{-i}$ such that (I) $|\{j \in N | j \neq i, u_j \notin U_\parallel\}| \leq \ell$, and (II) $\exists a^*$ s.t. $u_{1,a^*}(z_{a^*}) > u_{1,a}(z_a)$ for all $a \neq a^*$, we have $t_{i,a^*}(u_{-i}) = 0$, and $t_{i,a}(u_{-i}) = +\infty$ for all $a \neq a^*$.

We first observe that Lemma 5 implies G_0 . For u_{-1} s.t. $|\{i \in N | i \neq 1, u_i \notin U_\parallel\}| = 0$, $u_{-1} \in \tilde{\mathcal{U}}_{-1}$, thus part (i) of Lemma 5, agent-independence and the fact $k_i = 0$ for $i \neq 1$ imply that $\forall a \in A$:

$$t_{1,a}(u_{-1}) = \frac{1}{k_1} \max_{a' \in A} \left\{ \sum_{i \neq 1} k_i p_{i,a'} + C_{a'} \right\} - \frac{1}{k_1} \left(\sum_{i \neq 1} k_i p_{i,a} + C_a \right) = \frac{1}{k_1} \left(\max_{a' \in A} C_{a'} - C_a \right) = z_a.$$

We show in the following two steps that $G_\ell \Rightarrow H_\ell$ and $H_{\ell-1} \Rightarrow G_\ell$. This implies that G_ℓ holds for $\ell = n-1$, and completes the proof of the theorem. H_{n-1} also implies that when the dictator has a unique most preferred alternative at the fixed prices $\{z_a\}_{a \in A}$, the rest of the agents cannot be charged any payment.

Step 2.1: $G_\ell \Rightarrow H_\ell$ for $0 \leq \ell \leq n-1$.

W.l.o.g., consider agent $i = 2$, and some $u_{-2} \in \mathcal{U}_{-2}$ s.t. $|\{j \in N | j \neq 2, u_j \notin U_\parallel\}| = \ell$. For any $u_2 \in \tilde{U}_2$, in the economy (u_2, u_{-2}) , the number of agents in the profile u_{-1} with types outside of the parallel domain is at most ℓ . Since $|\{j \in N | j \neq 2, u_j \notin U_\parallel\}| = \ell$ and u_2 is parallel, $|\{j \in N | j \neq 1, u_j \notin U_\parallel\}| = \ell - 1$ if $u_1 \notin U_\parallel$, and $|\{j \in N | j \neq 1, u_j \notin U_\parallel\}| = \ell$ if $u_1 \in U_\parallel$. We know from G_ℓ that $t_{i,a}(u_{-1}) = z_a$ for all $a \in A$.

If there exists a unique agent-maximizing alternative for agent 1 given fixed prices $\vec{z}$, i.e. $\exists a^*$ s.t. $u_{1,a^*}(z_{a^*}) > u_{1,a}(z_a)$, $\forall a \neq a^*$, the only alternative that can be selected in the economy (u_2, u_{-2}) is a^* . This implies that for all $u_2 \in \tilde{U}_2$, $x(u_2, u_{-2}) = a^*$ must hold. Since $\tilde{U}_2$ is unrestricted, in order for a^* to be agent-maximizing for agent 2 for any $u_2 \in \tilde{U}_2$, we must have $t_{2,a}(u_{-2}) = \infty$ for all $a \neq a^*$. Since prices must be standard, $t_{2,a^*}(u_{-2}) = 0$ and this completes the proof of $G_\ell \Rightarrow H_\ell$.

Step 2.2: $H_{\ell-1} \Rightarrow G_\ell$ for all $1 \leq \ell \leq n-1$.

Let there be ℓ entries in u_{-1} that are outside of the parallel domain, and w.l.o.g. assume that $u_2, \dots, u_{\ell+1} \notin U_\parallel$. Assume that $\exists a \in A$ s.t. $t_{1,a}(u_{-1}) \neq z_a$, we first show a contradiction for the case that $t_{1,a}(u_{-1}) > z_a$, and then show that the other direction cannot hold either. First, note that it cannot be the case if $t_{1,a'}(u_{-1}) > z_{a'}$ for all $a' \in A$. This is because $\vec{z}$ as defined in (16) is a vector of standard prices with the minimal entry equal to 0. If $t_{1,a'}(u_{-1}) > z_{a'}$ for all $a' \in A$, we know $t_{1,a'}(u_{-1}) > 0$ for all a' , and this violates Lemma 1. W.l.o.g., we assume $t_{1,b}(u_{-1}) \leq z_b$.

Denote $\epsilon \triangleq t_{1,a}(u_{-1}) - z_a$, we know $\epsilon > 0$. Consider some parallel type of agent 1, $u_1 \in \tilde{U}_1$, where the willingness to pay is of the form:

$$\begin{aligned} p_{1,a} &= z_a + \epsilon/2, \\ p_{1,b} &= z_b + \epsilon/3, \\ p_{1,c} &= 0, \quad \forall c \neq a, b. \end{aligned}$$

This gives us $p_{1,b} - z_b = \epsilon/3$, $p_{1,c} - z_c = -z_c$, $\forall c \neq a, b$, and therefore $p_{1,a} - z_a = \epsilon/2 > p_{1,a'} - z_{a'}$ for all $a' \neq a$. Given Lemma 2, we know that a is the unique agent-maximizing alternative for agent 1 under the vector of prices $\vec{z}$.

In the economy $(u_1, u_2, \dots, u_{\ell+1}, u_{\ell+2}, \dots, u_n)$, there are $\ell - 1$ entries in the profile u_{-2} that are outside of the parallel domain: $|\{i \in N \mid i \neq 2, u_i \notin U_{\parallel}\}| = \ell - 1$. $H_{\ell-1}$ implies that $t_{2,a}(u_{-2}) = 0$, $t_{2,a'}(u_{-2}) = +\infty$ for all $a' \neq a$ thus a is the unique agent-maximizing alternative for agent 2. However, a cannot be agent-maximizing for agent 1, since

$$\begin{aligned} p_{1,a} - t_{1,a}(u_{-1}) - (p_{1,b} - t_{1,b}(u_{-1})) &\leq p_{1,a} - (z_a + \epsilon) - p_{1,b} + z_b \\ &= z_a + \epsilon/2 - (z_a + \epsilon) - (z_b + \epsilon/3) + z_b = -5/6\epsilon < 0. \end{aligned}$$

This contradicts DSIC, thus we conclude $t_{1,a}(u_{-1}) > z_a$ cannot be true. Similarly, if $t_{1,a}(u_{-1}) < z_a$, the price being standard requires that $z_a > t_{1,a}(u_{-1}) \geq 0$ and that there exists $b \in A$ s.t. $t_{1,b}(u_{-1}) \geq z_b$. Let $\epsilon \triangleq z_a - t_{1,a}(u_{-1}) > 0$, and let u_1 be a parallel type with willingness to pay

$$\begin{aligned} p_{1,a} &= z_a - \epsilon/2, \\ p_{1,b} &= z_b + \epsilon/3, \\ p_{1,c} &= 0, \quad \forall c \neq a, b. \end{aligned}$$

We can check that b is the unique agent-maximizing alternative for agent 1 under prices $\vec{z}$ thus $H_{\ell-1}$ implies that $t_{2,a}(u_{-2}) = \infty$, thus a cannot be agent-maximizing for agent 2 thus cannot be selected in the economy $(u_1, u_2, \dots, u_n)$. However, a is the unique agent-maximizing alternative for agent 1, since $p_{1,a} - t_{1,a}(u_{-1}) = \epsilon/2$, whereas $p_{1,b} - t_{1,b}(u_{-1}) \leq p_{1,b} - z_b = \epsilon/3$ and $p_{1,c} - t_{1,c}(u_{-1}) \leq 0$ for all $c \neq a, b$. This is a contradiction, thus $t_{1,a}(u_{-1}) = z_a$ must hold.

This completes the proof of this theorem. $\square$

From G_{n-1} and H_{n-1} , we know that when the dictator has a unique most preferred alternative under the fixed prices $\vec{a}$, the rest of the agents do not make any payment to the mechanism. However, when there are multiple most preferred alternatives that are tied for the dictator given the prices $\vec{z}$, H_{n-1} does not specify what must happen to the payments from the rest of the agents. We may consider various tie-breaking mechanisms among the alternatives toward which the dictator is indifferent, for example another fixed price dictatorship, or some generalized weighted VCG mechanism between two alternatives that are tied (in which case the general non-quasi-linear utility domain $\mathcal{U}^0$ is parallel). These mechanism would still satisfy (P1)-(P5), and would charge the non-dictators some non-zero payments in the degenerate case when the dictator is indifferent.

A.4 Relaxing IR and No Subsidy

Conditions (C1) and (C2) in Theorem 3 require the utility domain to deviate very minimally from the parallel domain, however, the negative result no longer holds if one of (P4) or (P5) is relaxed. (P4) IR and (P5) No subsidy require prices to be standard (Lemma 1). With standard prices, the shapes of an agent's utility functions where the prices are negative, or where the utilities are below $\min_{a \in A} v_{i,a}$, are irrelevant to which alternative is agent-maximizing for this agent. The parallel domain only requires that the utility functions in the range that is relevant to be horizontal translations of each other.

As we have shown in the proof of Lemma 5, an affine maximizer as the choice rule together with DSIC determine the agent-independent prices that each agent faces up to a constant (when there are no ties). The requirement that prices being standard then fully pins down the agent-independent prices. Without (P4), (P5), a local violation of “the relative willingness to pay between any two alternatives remains constant” (e.g. u_2^* as in Figure 2) can be made irrelevant by setting non-standard prices carefully and we still get an incentive compatible mechanism.

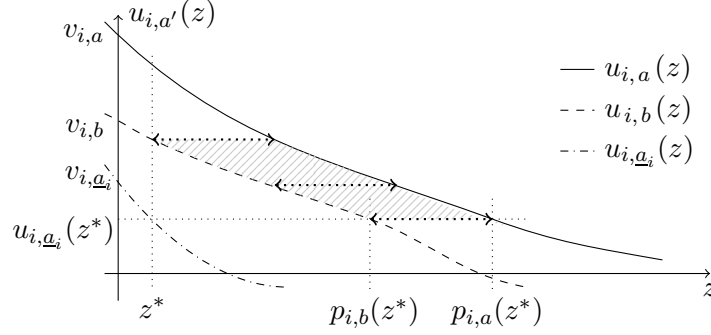


Figure 4: An example type in the parallel domain w.r.t. z^* .

Here we provide a family of such mechanisms that violates only one of (P4) or (P5). This shows that the parallel domain as we defined in Section 2 is not the only maximal utility domain where mechanisms that satisfy (P1)-(P3) exist. We say a utility domain is parallel w.r.t. price z^* if the relative willingness to pay remains the same in the range where (I) payments are at least z^* , and (II) the utilities are weakly above $\min_{a' \in A} \{u_{i,a'}(z^*)\}$, as illustrated in Figure 4. Denote $p_{i,a}(z^*) \triangleq u_{i,a}^{-1}(\min_{a' \in A} \{u_{i,a'}(z^*)\})$.

Definition 7. A utility domain for an agent $U_i \subset U^0$ is *parallel with respect to $z^* \in \mathbb{R}$* if $\forall u_i \in U_i$, $\forall a, b \in A$ s.t. $u_{i,a}(z^*) \geq u_{i,b}(z^*)$, and $\forall z \in [z^*, p_{i,b}(z^*)]$, $u_{i,a}(z + (p_{i,a}(z^*) - p_{i,b}(z^*))) = u_{i,b}(z)$.

Definition 8. The generalized weighted VCG mechanism with fixed payment z^* parametrized by z^* , non-negative weights $\{k_i\}_{i \in N}$ and real constants $\{C_a\}_{a \in A}$ collects type profile $\hat{u}$ from the agents, and computes the willingness to pay $\{\hat{p}_{i,a}(z^*)\}_{i \in N, a \in A}$ w.r.t. z^* .

- Choice rule: $x(\hat{u}) = a^*$ where $a^* \in \arg \max_{a \in A} \{\sum_{i \in N} k_i \hat{p}_{i,a}(z^*) + C_a\}$, breaking ties arbitrarily.
- Payment rule: $t_i(\hat{u}) = z^*$ for $i \in N$ s.t. $k_i = 0$; for i s.t. $k_i \neq 0$:

$$t_i(\hat{u}) = \frac{1}{k_i} \left(\sum_{j \neq i} k_j \hat{p}_{j,a_{-i}^*}(z^*) + C_{a_{-i}^*} - \sum_{j \neq i} k_j \hat{p}_{j,a^*}(z^*) - C_{a^*} \right) + z^*,$$

where $a_{-i}^* \in \arg \max_{a \in A} \{\sum_{j \neq i} k_j \hat{p}_{j,a}(z^*) + C_a\}$.

Proposition 1. Assuming U_i is parallel w.r.t. z^* for all $i \in N$, the generalized weighted VCG mechanism with fixed payment z^* and at least one non-zero coefficient k_i satisfies (P1)-(P3). In addition, (P4) is satisfied if z^* is non-positive, whereas (P5) is satisfied if z^* is non-negative.

Proof. It is immediate that the mechanism is DSIC for agents $i \in N$ s.t. $k_i = 0$. Fix an agent $i \in N$ s.t. $k_i > 0$. Her agent-independent prices is given by: $t_{i,a}(\hat{u}_{-i}) = 1/k_i (\sum_{j \neq i} k_j \hat{p}_{j,a_{-i}^*}(z^*) + C_{a_{-i}^*} - \sum_{j \neq i} k_j \hat{p}_{j,a}(z^*) - C_a) + z^*$, which implies $t_{i,a_{-i}^*}(\hat{u}_{-i}) = z^*$ and $t_{i,a}(\hat{u}_{-i}) \geq z^*$ for all a . Similar to Lemma 1, we can show that the agent-maximizing alternative for agent i given $\{t_{i,a}(\hat{u}_{-i})\}_{a \in A}$ is the maximizer of $\hat{p}_{i,a}(z^*) - t_{i,a}(\hat{u}_{-i})$. We can then examine that a^* is a maximizer of $\hat{p}_{i,a}(z^*) - t_{i,a}(\hat{u}_{-i})$, and this implies that the choice rule is agent-maximizing and this completes the proof of DSIC.

When $z^* \leq 0$, the minimum agent-independent price for agent i among all alternatives is non-positive, thus the agent-maximizing alternative gives the agent a utility at least $\min_{a \in A} \hat{u}_{i,a}(z^*) \geq \min_{a \in A} \hat{u}_{i,a}(0)$ thus (P4) IR is satisfied. When $z^* \geq 0$, all agent-independent prices are non-negative, thus the mechanism satisfies (P5) No subsidy. $\square$

A.4.1 Proof of Theorem 4

Theorem 4. *With at least three alternatives and a utility domain $\mathcal{U} = \prod_{i=1}^n U_i$ s.t. for each $i \in N$, U_i contains an unrestricted linear domain with two slopes, a social choice mechanism satisfying (P1)-(P3) must be a fixed price dictatorship.*

We first define *strictly parallel domains* as sets of agent types where utility curves are horizontal shifts of each other *everywhere*. Quasi-linear utility functions, for example, are strictly parallel.

Definition 9 (Strictly Parallel Domain). A utility domain $U_i \subset U^0$ is a *strictly parallel domain* if for all $u_i \in U_i$,

$$u_{i,a}(z + (p_{i,a} - p_{i,b})) = u_{i,b}(z), \quad \forall z \in \mathbb{R}, \quad \forall a, b \in A. \quad (17)$$

On any strictly parallel domain, $p_{i,a} - t_{i,a} \geq p_{i,b} - t_{i,b} \Leftrightarrow u_{i,a}(t_{i,a}) \geq u_{i,b}(t_{i,b})$ holds for any prices $t_{i,a}, t_{i,b} \in \mathbb{R}$, therefore, $\arg \max_{a \in A} \{u_{i,a}(t_{i,a})\} = \arg \max_{a \in A} \{p_{i,a} - t_{i,a}\}$ holds for any prices, without the requirement that the prices are standard. This implies that W-Mon is a necessary condition for any mechanism that is deterministic and DSIC for strictly parallel utility domains, and that Roberts' theorem can be generalized without the additional assumptions (P4) and (P5).

Lemma 7. With any strictly parallel utility domain $\mathcal{U}$, every social choice mechanism that is DSIC and deterministic must satisfy W-Mon in terms of willingness to pay.

Lemma 8 (Roberts' Theorem on Strictly Parallel Domains). With three or more alternatives, and an unconstrained strictly parallel domain $\mathcal{U}$, for every social choice mechanism that satisfies (P1)-(P3), there exist non-negative weights $k_1, \dots, k_n$ (not all of them zero) and constants $C_1, \dots, C_m$ such that for all $u \in \mathcal{U}$, $x(u) \in \arg \max_{a \in A} \{\sum_{i=1}^n k_i p_{i,a} + C_a\}$.

We now prove the impossibility result for the linear domain with two slopes.

Proof of Theorem 4. Let $U_i^\alpha \subset U_i$ be the set of all $u_i \in U_i$ s.t. $u_{i,a}(z) = v_{i,a} - \alpha_i z$ holds for all $a \in A$, i.e. the set of " α types". We know U_i^α is a strictly parallel domain with unrestricted willingness to pay. Let $\mathcal{U}^\alpha \triangleq \prod_{i=1}^n U_i^\alpha$, Lemma 6 implies that fixing any mechanism (x, t) on $\mathcal{U}$ that satisfies (P1)-(P3), the restriction of (x, t) on $\mathcal{U}^\alpha$ must also satisfy (P1)-(P3). Lemma 8 then implies that there exists non-negative coefficients $\{k_i\}_{i \in N}$ and real constants $\{C_a\}_{a \in A}$ s.t. $k_i \neq 0$ for some $i \in N$ and $x(u) \in \arg \max_{a \in A} \{\sum_{i=1}^n k_i p_{i,a} + C_a\}$ for all $u \in \mathcal{U}^\alpha$. With the same arguments as in the proof of Lemma 5, we can show that $\forall i \in N$ s.t. $k_i > 0$, $\forall a, b \in A$, the difference in the agent-independent prices satisfies:

$$t_{i,a}(u_{-i}) - t_{i,b}(u_{-i}) = \frac{1}{k_i} \left(\sum_{j \neq i} k_j p_{j,b} + C_b - \sum_{j \neq i} k_j p_{j,a} - C_a \right). \quad (18)$$

Assume without loss of generality that $k_1 > 0$, we prove that agent 1 is the fixed-price dictator with the following steps.

- Step 1. $k_i = 0$ for all $i \neq 1$.
- Step 2. There exist fixed prices $\vec{z} \in \mathbb{R}^m$ s.t. $\forall u_{-1} \in \mathcal{U}_{-1}^\alpha, \forall a \in A, t_{1,a}(u_{-1}) = z_a$.
- Step 3. Agent 1 is the fixed-price dictator for any $u \in \mathcal{U}$.

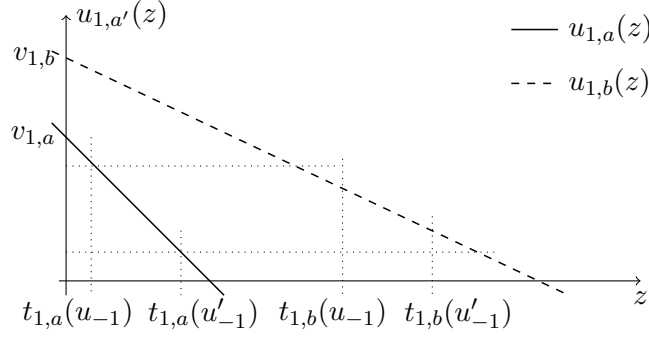


Figure 5: Illustration of $u_1 \in U_1 \setminus U_1^\alpha$, for the proof of Claim 4.

The reason that Step 2 is not immediately implied by Step 1 is that without the assumptions of (P4) and (P5), so that prices are not necessarily standard, (18) only pins down the agent-independent prices up to a constant, and we need to show by induction that the prices $t_{1,a}(u_{-1})$ must be fixed for all $u_{-1} \in \mathcal{U}_{-1}^\alpha$. We then prove in Step 3 by induction that $t_{1,a}(u_{-1}) = z_a$ must hold for all $u_{-1} \in \mathcal{U}_{-1}$, which shows that agent 1 is the fixed price dictator.

Step 1. We show by contradiction that $k_2 = 0$ must hold. The same argument can be repeated for all $i \neq 1$. We first prove the following claim.

Claim 4. Fix any $u_{-1,-2} \in \mathcal{U}_{-1,-2}^\alpha$, and assume $k_1, k_2 > 0$. $\forall a, b \in A$, and $\forall u_2, u'_2 \in U_2^\alpha$, $p_{2,a} - p_{2,b} = p'_{2,a} - p'_{2,b} \Rightarrow t_{1,a}(u_2, u_{-1,-2}) = t_{1,a}(u'_2, u_{-1,-2})$.

Proof. Let $u_2, u'_2 \in U_2^\alpha$ be two types of agent 2 s.t. $p_{2,a} - p_{2,b} = p'_{2,a} - p'_{2,b}$, and assume for contradiction that $t_{1,a}(u_2, u_{-1,-2}) \neq t_{1,a}(u'_2, u_{-1,-2})$. Denote $u_{-1} \triangleq (u_2, u_{-1,-2})$ and $u'_{-1} \triangleq (u'_2, u_{-1,-2})$. $t_{1,a}(u_{-1}) - t_{1,b}(u_{-1}) = t_{1,a}(u'_{-1}) - t_{1,b}(u'_{-1})$ follows from (18). Assume w.l.o.g. that $t_{1,a}(u_{-1}) < t_{1,a}(u'_{-1})$, and that $\alpha_1 > \beta_1$. We can construct $u_1 \in U_1$ as illustrated in Figure 5 such that $u_{1,a}(z) = v_{1,a} - \alpha_1 z$, $u_{1,b}(z) = v_{1,b} - \beta_1 z$, and that

$$u_{1,c}(t_{1,c}(u_{-1})) < u_{1,b}(t_{1,b}(u_{-1})) < u_{1,a}(t_{1,a}(u_{-1})), \quad \forall c \neq a, b, \quad (19)$$

$$u_{1,c}(t_{1,c}(u'_{-1})) < u_{1,a}(t_{1,a}(u'_{-1})) < u_{1,b}(t_{1,b}(u'_{-1})), \quad \forall c \neq a, b. \quad (20)$$

Such u_1 exists since $\alpha_1 > \beta_1$, $t_{1,a}(u_{-1}) - t_{1,b}(u_{-1}) = t_{1,a}(u'_{-1}) - t_{1,b}(u'_{-1})$, and the values are unrestricted. Agent-maximizing for agent 1 then implies $x(u_1, u_{-1}) = a$ and $x(u_1, u'_{-1}) = b$. Note that the agent-independent prices facing agent 2, $\{t_{2,a'}(u_{-2})\}_{a' \in A}$, remain the same in the two economies. Agent-maximization for agent 2 therefore implies

$$\begin{aligned} p_{2,a} - t_{2,a}(u_{-2}) &\geq p_{2,b} - t_{2,b}(u_{-2}), \\ p'_{2,a} - t_{2,a}(u_{-2}) &\leq p'_{2,b} - t_{2,b}(u_{-2}). \end{aligned}$$

Given the assumption $p_{2,a} - p_{2,b} = p'_{2,a} - p'_{2,b}$, we must have

$$p_{2,a} - p_{2,b} = p'_{2,a} - p'_{2,b} = t_{2,a}(u_{-2}) - t_{2,b}(u_{-2}). \quad (21)$$

Let ϵ be some small positive number such that $0 < \epsilon < u_{1,b}(t_{1,b}(u'_{-1})) - u_{1,a}(t_{1,a}(u'_{-1}))$. Consider type $u'_1 \in U_1$ such that for all $z \in \mathbb{R}$, $u'_{1,a}(z) = u_{1,a}(z)$, $u'_{1,b}(z) = u_{1,b}(z) - \epsilon$ and $u'_{1,c}(z) = u_{1,c}(z)$. In words, u'_1 is identical to u_1 , except that $v'_{1,b} = v_{1,b} - \epsilon$, where ϵ is small enough that both (19) and (20)

still hold if u_1 is replaced with u'_1 . Thus we have $x((u'_1, u_2, u_{-1,-2})) = a$ and $x((u'_1, u'_2, u_{-1,-2})) = b$ from agent-maximization for agent 1 given u'_1 .

Replacing u_1 with u'_1 results in a decrease in the willingness to pay for alternative b , thus $p'_{1,a} - p'_{1,b} > p_{1,a} - p_{1,b}$. Given (18) and the assumption that $k_1, k_2 > 0$, we know $t_{2,a}((u'_1, u_{-1,-2})) - t_{2,b}((u'_1, u_{-1,-2})) < t_{2,a}(u_{-2}) - t_{2,b}(u_{-2})$. Combined with (21), we know:

$$\begin{aligned} t_{2,a}((u'_1, u_{-1,-2})) - t_{2,b}((u'_1, u_{-1,-2})) &< p'_{2,a} - p'_{2,b} \\ \Rightarrow p'_{2,a} - t_{2,a}((u'_1, u_{-1,-2})) &> p'_{2,b} - t_{2,b}((u'_1, u_{-1,-2})), \end{aligned}$$

meaning that the alternative $x((u'_1, u'_2, u_{-1,-2})) = b$ is not agent-maximizing for agent 2 in economy $(u'_1, u'_2, u_{-1,-2})$. This contradicts DSIC, thus we conclude that $t_{1,a}(u_2, u_{-1,-2}) = t_{1,a}(u'_2, u_{-1,-2})$ must hold. $\square$

Assume $k_1, k_2 > 0$, fix any $u_{-1,-2} \in \mathcal{U}_{-1,-2}^\alpha$ and some $u_2^* \in U_2^\alpha$, we know $t_{1,a}(u_2^*, u_{-1,-2}) < \infty$, since $k_1 > 0$ and U_1^α is unrestricted thus there exists $u_1 \in U_1^\alpha$ s.t. $x((u_1, u_2^*, u_{-1,-2})) = a$. Denote $\Delta_{a,b}^* \triangleq p_{2,a}^* - p_{2,b}^*$, $\Delta_{a,c}^* \triangleq p_{2,a}^* - p_{2,c}^*$ and $z_a \triangleq t_{1,a}(u_2^*, u_{-1,-2})$. We know from Claim 4 that $\forall u_2 \in U_2^\alpha$ s.t. $p_{2,a} - p_{2,c} = \Delta_{a,c}^*$, $t_{1,a}(u_2, u_{-1,-2}) = z_a$. For any $\Delta_{a,b} \in \mathbb{R}$, we can find $u'_2 \in U_2^\alpha$ s.t. $p'_{2,a} - p'_{2,b} = \Delta_{a,b}$ and $p'_{2,a} - p'_{2,c} = \Delta_{a,c}^*$, for which $t_{1,a}(u'_2, u_{-1,-2}) = z_a$. Apply Claim 4 again, we know that for all $u_2 \in U_2^\alpha$ s.t. $p_{2,a} - p_{2,b} = \Delta_{a,b}$, $t_{1,a}(u_2, u_{-1,-2}) = z_a$. Since this holds for all $\Delta_{a,b} \in \mathbb{R}$, we conclude that $\forall u_2 \in U_2^\alpha$, $t_{1,a}(u_2, u_{-1,-2}) = z_a$. This implies that even for $u_2, u'_2 \in U_2^\alpha$ s.t. $p_{2,a} - p_{2,b} \neq p'_{2,a} - p'_{2,b}$, we still have $t_{1,a}(u_2, u_{-1,-2}) - t_{1,b}(u_2, u_{-1,-2}) = t_{1,a}(u'_2, u_{-1,-2}) - t_{1,b}(u'_2, u_{-1,-2})$. This contradicts (18), thus $k_2 = 0$ must hold.

Step 2. Given (18) and Step 1 that $k_i = 0$ for all $i \neq 1$, we know that for all $u_{-1} \in \mathcal{U}_{-1}^\alpha$, for any pair of alternatives $a, b \in A$, $t_{1,a}(u_{-1}) - t_{1,b}(u_{-1}) = 1/k_1(C_b - C_a)$ must hold. Denote this price difference as $\delta_{a,b} \triangleq 1/k_1(C_b - C_a)$. We prove the following claim.

Claim 5. For all $i \neq 1$, for all $u_{-1,-i} \in \mathcal{U}_{-1,-i}^\alpha$, and for all $u_i, u'_i \in U_i^\alpha$, if there exists $a^* \in A$ s.t. $t_{1,a^*}((u'_i, u_{-1,-i})) \neq t_{1,a^*}((u_i, u_{-1,-i}))$, then $p_{i,a} = p'_{i,a}$ for all $a \in A$, i.e. u_i and u'_i have the same willingness to pay for all $a \in A$.

Proof. Fix agent $i = 2$ and any $u_{-1,-i} \in \mathcal{U}_{-1,-i}^\alpha$, denote $u_{-1} \triangleq (u_2, u_{-1,-2})$, $u'_{-1} = (u'_2, u_{-1,-2})$, and assume that there exists $a \in A$ s.t. $t_{1,a}(u'_{-1}) > t_{1,a}(u_{-1})$. We first prove $p_{2,a} - p'_{2,a} = p_{2,b} - p'_{2,b}$ for all $b \in A$. This pins down the willingness to pay for all alternatives up to a constant. The fact that the smallest willingness to pay among all alternatives must be zero then implies $p_{2,b} = p'_{2,b}$ for all $b \in A$. Repeating the same arguments for all $i \neq 1$ completes the proof of this claim.

We now prove $p_{2,a} - p'_{2,a} = p_{2,b} - p'_{2,b}$. First, we know $t_{1,a}(u'_{-1}) - t_{1,b}(u'_{-1}) = t_{1,a}(u_{-1}) - t_{1,b}(u_{-1}) = \delta_{a,b}$ from Step 1. Same as the proof of Claim 4, we may find $u_1 \in U_1$ as shown in Figure 5, where (19) and (20) both hold. Agent-maximization for agent 1 then implies

$$x((u_1, u_2, u_{-1,-2})) = a, \quad x((u_1, u'_2, u_{-1,-2})) = b. \quad (22)$$

Similarly, there exists $u'_1 \in U_1$ as illustrated in Figure 6 such that $u'_{1,a}(z) = v'_{1,a} - \beta_1 z$, $u'_{1,b}(z) = v'_{1,b} - \alpha_1 z$, and that

$$\begin{aligned} u'_{1,c}(t_{1,c}(u_{-1})) &< u'_{1,a}(t_{1,a}(u_{-1})) < u'_{1,b}(t_{1,b}(u_{-1})), \quad \forall c \neq a, b, \\ u'_{1,c}(t_{1,c}(u'_{-1})) &< u'_{1,b}(t_{1,b}(u'_{-1})) < u'_{1,a}(t_{1,a}(u'_{-1})), \quad \forall c \neq a, b. \end{aligned}$$

Agent-maximization for agent 1 with type u'_1 then requires

$$x((u'_1, u_2, u_{-1,-2})) = b, \quad x((u'_1, u'_2, u_{-1,-2})) = a. \quad (23)$$

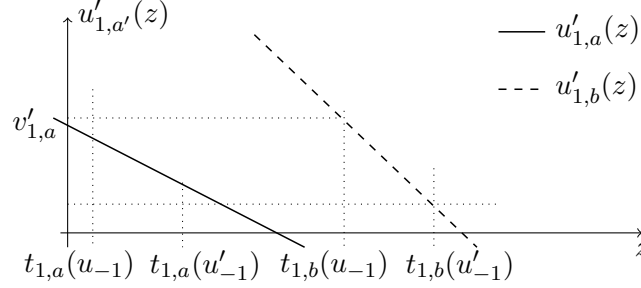


Figure 6: Illustration of $u'_1 \in U_1$, for the proof of Claim 5.

Given (22), we know from W-Mon that $p_{2,a} - p'_{2,a} \geq p_{2,b} - p'_{2,b}$ must hold. Similarly, we get $p_{2,a} - p'_{2,a} \leq p_{2,b} - p'_{2,b}$ from (23), therefore $p_{2,a} - p'_{2,a} = p_{2,b} - p'_{2,b}$. $\square$

Intuitively, Claim 5 shows that for any $i \neq 1$, for any $u_{-1,-i} \in \mathcal{U}_{-1,-i}^\alpha$, and any all $u_i, u'_i \in U_i^\alpha$, if there exists $a \in A$ s.t. $p_{i,a} \neq p'_{i,a}$, then we must have $t_{1,a'}((u'_i, u_{-1,-i})) = t_{1,a'}((u_i, u_{-1,-i}))$ for all $a' \in A$. Fix any $u_{-1}^* = (u_2^*, \dots, u_n^*) \in \mathcal{U}_{-1}^\alpha$, and define $z_a \triangleq t_{1,a}(u_{-1}^*)$ for all $a \in A$. We prove by induction that $\forall u_{-1} \in \mathcal{U}_{-1}^\alpha$, $t_{1,a}(u_{-1}) = z_a$ must hold for all $a \in A$. For any $\ell = 0, 1, \dots, n-1$, let the induction statements be

$$G_\ell^*: \forall u_{-1} \in \mathcal{U}_{-1}^\alpha \text{ s.t. } |\{i \in N | i \neq 1, u_i \neq u_i^*\}| \leq \ell, t_{1,a}(u_{-1}) = z_a, \forall a \in A.$$

$$H_\ell^*: \forall i \neq 1, \text{ for all } u_1 \in U_1 \text{ and } \forall u_{-1,-i} \in \mathcal{U}_{-1,-i}^\alpha, \text{ if (I) } |\{j \in N | j \neq 1, j \neq i, u_j \neq u_j^*\}| \leq \ell, \text{ and (II) } \exists a^* \text{ s.t. } u_{1,a^*}(z_{a^*}) > u_{1,a}(z_a), \forall a \neq a^*, \text{ then } t_{i,a}(u_{-i}) = +\infty \forall a \neq a^*.$$

G_0^* trivially holds from agent-independence since when $|\{i \in N | i \neq 1, u_i \neq u_i^*\}| \leq 0$, $u_{-1} = u_{-1}^*$. We now prove $G_\ell^* \Rightarrow H_\ell^*$ and $H_{\ell-1}^* \Rightarrow G_\ell^*$, and this would complete the proof of the claim of this step, which is G_{n-1}^* .

Step 2.1. $G_\ell^ \Rightarrow H_\ell^*$ for all $0 \leq \ell \leq n-1$.*

Consider agent $i = 2$, and fix any $u_{-1,-2} \in \mathcal{U}_{-1,-2}^\alpha$ s.t. $|\{j \in N | j \neq 1, j \neq 2, u_j \neq u_j^*\}| \leq \ell$. We know from G_ℓ^* that for all $a \in A$, $t_{1,a}(u_2^*, u_{-1,-2}) = z_a$. Fix some $u_1 \in U_1$ for which there exists $a^* \in A$ s.t. $u_{1,a^*}(z_{a^*}) > u_{1,a}(z_a)$ for all $a \neq a^*$. For any alternative $b \neq a^*$, and any $u_2 \in U_2^\alpha$ s.t. $p_{2,b} > p_{2,b}^*$, we know from Claim 5 that $t_{1,a}(u_2, u_{-1,-2}) = t_{1,a}(u_2^*, u_{-1,-2}) = z_a$ for all $a \in A$, thus a^* is also the unique agent-maximizing alternative for agent 1 in the economy $(u_1, u_2, u_{-1,-2})$. Since $p_{2,b}$ can be arbitrarily large, we must have $t_{2,b}(u_1, u_{-1,-2}) = \infty$ so that a^* is also always agent-maximizing for agent 2. The same argument can be repeated for all agents $i \neq 1$.

Step 2.2. $G_{\ell-1}^$ and $H_{\ell-1}^* \Rightarrow G_\ell^*$, for all $1 \leq \ell \leq n-1$.*

Consider any type profile $u_{-1} \in \mathcal{U}_{-1}^\alpha$ such that $|\{i \in N | i \neq 1, u_i \neq u_i^*\}| = \ell$. We assume w.l.o.g. that $u_2 \neq u_2^*$. We know from $G_{\ell-1}^*$ that $t_{1,a}(u_2^*, u_{-1,-2}) = z_a$ holds for all $a \in A$, since in $(u_2^*, u_{-1,-2})$, $|\{i \in A | i \neq 1, u_i \neq u_i^*\}| = \ell-1$. Given that both u_2 and u_2^* are in U_2^α , if there exists any alternative $a \in A$ s.t. $p_{2,a} \neq p_{2,a}^*$, we know from Claim 5 that $t_{1,a}(u_2, u_{-1,-2}) = t_{1,a}(u_2^*, u_{-1,-2}) = z_a$ holds for all $a \in A$, which is what we are looking for. Therefore, the only remaining case is for $u_2 \in U_2^\alpha$ s.t. $p_{2,a} = p_{2,a}^*$ for all $a \in A$.

Assume toward a contradiction, that there exists $u_2 \in U_2^\alpha$ s.t. $p_{2,a} = p_{2,a}^*$ for all $a \in A$, for which there exists an alternative, say alternative a s.t. $t_{1,a}(u_2, u_{-1,-2}) \neq t_{1,a}(u_2^*, u_{-1,-2}) = z_a$. Assume $t_{1,a}(u_2, u_{-1,-2}) > t_{1,a}(u_2^*, u_{-1,-2})$, and the other direction can be proved similarly. Fix any alternative $b \neq a$, we know from (18) that $t_{1,a}(u_2, u_{-1,-2}) - t_{1,b}(u_2, u_{-1,-2}) = t_{1,a}(u_2^*, u_{-1,-2}) -$

$t_{1,b}(u_2^*, u_{-1,-2})$ must hold. Similar to the proof of Claim 4, we can find $u_1 \in U_1$ s.t. $u_{1,a}(z) = v_{1,a} - \alpha_1 z$, $u_{1,b} = v_{1,b} - \beta_1 z$ for all $z \in \mathbb{R}$, and that

$$\begin{aligned} u_{1,c}(t_{1,c}(u_2^*, u_{-1,-2})) &< u_{1,b}(t_{1,b}(u_2^*, u_{-1,-2})) < u_{1,a}(t_{1,a}(u_2^*, u_{-1,-2})), \quad \forall c \neq a, b, \\ u_{1,c}(t_{1,c}(u_2, u_{-1,-2})) &< u_{1,a}(t_{1,a}(u_2, u_{-1,-2})) < u_{1,b}(t_{1,b}(u_2, u_{-1,-2})), \quad \forall c \neq a, b. \end{aligned}$$

We know that in the economy $(u_1, u_2, u_{-1,-2})$, alternative b is the unique agent-maximizing alternative for agent 1. However, given since $t_{1,a'}(u_2^*, u_{-1,-2}) = z_{a'}$ holds for all $a' \in A$, we know that given the vector of prices $\vec{z}$, alternative a is the unique agent-maximizing alternative for agent 1. Therefore, $H_{\ell-1}^*$ together with $|\{i \in A | i \neq 1, i \neq 2, u_i \neq u_i^*\}| = \ell - 1$ implies that $t_{2,b}(u_1, u_{-1,-2}) = \infty$. This shows that alternative b cannot be agent-maximizing for agent 2 in the economy $(u_1, u_2, u_{-1,-2})$. This violates DSIC, completes the proof of $G_{\ell-1}^*$ and $H_{\ell-1}^* \Rightarrow G_\ell^*$, and also the proof of Step 2 of this theorem.

Step 3. Step 2 implies that when $u_{-1} \in \mathcal{U}_{-1}^\alpha$, the outcome of the mechanism must be determined according to the fixed price dictator where agent 1 is the dictator and the fixed prices are given by $\vec{z}$. The proof of the third step is very similar to the proof of Step 2 of Theorem 3: by induction on the number of agents whose type is outside of U_i^α , we can show that the outcome must be determined by the same fixed-price dictatorship for any $u \in \mathcal{U}$. This completes the proof of Theorem 4. $\square$